Some Fixed Point Theorems for Contractive Type Mapping in n-banach spaces

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Research Article

Abstract: Some Fixed Point Theorems for a class of mappings with contractive iterates in a setting of *n*-Banach spaces have been proved.

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1. Introduction

The concept of 2-normed spaces was introduced and studied by German Mathematician Siegfried Gähler [11], [12], [13], [14] in a series of paper as appeared in 1960's. Later on Misiak [1] had also developed the notion of an *n*-norm in 1989. The concept on *n*-inner product spaces is also due to Misiak who had studied the same as early as 1980 , see [1]. Following this one sees systematic development in theory of linear *n*-normed spaces as made by S. S. Kim and Y. J. Cho [10], R. Maleceski [7] and H. Gunawan and Mashadi [5]. H. Dutta and B. Surendra Reddy in [6], have shown that under certain cases convergence and completion in a *n*-normed space is equivalent to those in $(n - r)$ norm, $r = 1, 2, \ldots, n - 1$. For related works of *n*-metric spaces and *n*-inner product spaces one may see [1], [2] and [3].

Recently H. Gunawan and M. Mashadi in [5] have proved some fixed point theorems for contractive mappings acting over a finite dimensional *n*-Banach space. In this paper we also present some fixed point theorems for mappings with contractive iterates supposed to act on any *n*-Banach space.

2. We recall some preliminary Definitions related to our findings as presented here below.

Definition 2.1: Given a natural number *n*, let *X* be a real vector space of dimension $d \geq n$ (*d* may be infinity). A real valued function $\|.,\ldots,.\|$ on X^n satisfying following four properties,

(i) $\|x_1,\ldots,x_n\| = 0$ if and only if $x_1,\ldots,x_n$ are linearly dependent in *X*.

(ii) $\|x_1,\ldots,x_n\|$ is invariant under permutation of $x_1, x_2,\ldots,x_n$,

(iii) $\|x_1,\ldots,x_{n-1},\alpha x_n\| = |\alpha| \|x_1,\ldots,x_{n-1},x_n\|$ for every $\alpha \in \mathbb{R}$

(iv) $\|x_1,\ldots,x_{n-1},y+z\| \leq \|x_1,\ldots,x_{n-1},y\| + \|x_1,\ldots,x_{n-1},z\|$ for all *y* and *z* in *X*, is called an *n*-norm over *X* and the pair $\left(X, \underbrace{\|.,\ldots,.\|}_{n\text{ places}} \right)$ is called an *n*-normed spaces.

Definition 2.2.: A sequence x_k in an *n*-normed space $X, \|.,\ldots,.\|$ is said to converge to an element $x \in X$ (in the *n*-norm) whenever $\lim_{k\to\infty} \|u_1,\ldots,u_{n-1},x_k - x\| = 0$ for every $u_1,\ldots,u_{n-1} \in X$.

Definition 2.3.: A sequence x_k in an *n*-normed space $X, \|.,\ldots,.\|$ is said to be a Cauchy sequence with respect to *n*-norm if

$\lim\limits_{k,l\to\infty}\|u_1,...,u_{n-1},x_k-x_l\|=0$ for every $u_1,...,u_{n-1}\in X$.

Definition 2.4.: If every Cauchy sequence in *X* converges to an element $x\in X$, then *X* is said to be complete (with respect to the *n*-norm).
A complete *n*-normed space is called an *n*-Banach space.

Definition 2.5.: Let *X* be a *n*-Banach space and *T* be a self mapping of *X*. *T* is said to be continuous at *x* if for every sequence x_k in *X*, $x_k \to x$ as $k\to\infty$ implies $T\ \ x_k \ \ \to T\ \ x$ as $k\to\infty$ in *X*.

Example 2.1: Take the function space $L_2\ \ 0,1$ of all square integrable functions over the closed interval [0,1]. Then $p^{\,0}, p^{\,1}, p^{\,2},..., p^{\,j},...$ where $p^{\,j}$'s are polynomials with $p^{\,j}\ \ t\ \ = t^j, 0\le t\le 1$ and $j=0,1,2,...$ becomes a linearly independent set in $L_2\ \ 0,1$. For a natural number *n* define an *n*-norm over $L_2\ \ 0,1$ as $L_2\ \ 0,1\times L_2\ \ 0,1\times ...\times L_2\ \ 0,1$ (*n* factors) $\to$ Reals denoted by $\|f_1, f_2,..., f_n\|$ and given by

$$\|f_1, f_2,..., f_n\| = |\begin{bmatrix} \int_0^1 f_1 p^{\,0}\,dt & \int_0^1 f_1 p^{\,1}\,dt & ... & \int_0^1 f_1 p^{\,n}\,dt \\ ... & ... & ... & ... \\ \int_0^1 f_n p^{\,0}\,dt & \int_0^1 f_n p^{\,1}\,dt & ... & \int_0^1 f_n p^{\,n}\,dt \end{bmatrix}|$$

, where $f_1, f_2,..., f_n\in L_2\ \ 0,1$.

Note that in an *n*-normed space $X, \|.,...,.\|$ we have for instance $\|x_1,....,x_n\|\ge 0$ and $\|x_1,...,x_{n-1},x_n\| = \|x_1,...,x_{n-1},x_n+\alpha_1 x_1+...+\alpha_{n-1}x_{n-1}\|$ for all $x_1,x_2,...,x_n\in X$ and for all scalars $\alpha_1,...\alpha_{n-1}\in$ □ . Then it is a routine argument to see that $L_2\ \ 0,1$ is an *n*-normed Banach space (see section 2.1 of [5]).

3. Theorem 3.1

Let *T* be a self-mapping of *X* such that there exists *h* where $0<h<\frac{1}{2}$ and for all $x,y,u_1,...,u_{n-1}\in X$

$$\|T\ \ x\ \ -T\ \ y\ \ ,u_1,...,u_{n-1}\| \le h.\max\ \ \|x-y,u_1,...,u_{n-1}\|,$$
$$\|x-T\ \ x\ \ ,u_1,...,u_{n-1}\|,\|y-T\ \ y\ \ ,u_1,...,u_{n-1}\|,\|x-T\ \ y\ \ ,u_1,...,u_{n-1}\|,\|y-T\ \ x\ \ ,u_1,...,u_{n-1}\| \qquad (3.1)$$

then *T* has a unique fixed point *z* in *X* with $\lim\limits_{k\to\infty} T^k\ \ x_0\ \ = z$ for each $x_0\in X$.

Proof: Let $x_0\in X$, and define a recursive sequence x_k by

$$x_{k+1} = T\ \ x_k\ \ = T^{k+1}\ \ x_0\ \ , k=0,1,2,....$$

Then by (3.1) we have,

$$\|x_k-x_{k+1},u_1,...,u_{n-1}\| = \|T\ \ x_{k-1}\ \ -T\ \ x_k\ \ ,u_1,...,u_{n-1}\|$$
$$\le h.\max\ \ \|x_{k-1}-x_k,u_1,...,u_{n-1}\|,\|x_{k-1}-x_k,u_1,...,u_{n-1}\|,\|x_k-x_{k+1},u_1,...,u_{n-1}\|,$$
$$\|x_{k-1}-x_{k+1},u_1,...,u_{n-1}\|,\|x_k-x_k,u_1,...,u_{n-1}\|$$
$$= h.\max\ \ \|x_{k-1}-x_k,u_1,...,u_{n-1}\|,\|x_k-x_{k+1},u_1,...,u_{n-1}\|,\|x_{k-1}-x_{k+1},u_1,...,u_{n-1}\|\ \ .$$

Now $\|x_k - x_{k+1}, u_1, ..., u_{n-1}\| \leq h.\|x_k - x_{k+1}, u_1, ..., u_{n-1}\|$ is impossible, because $0 < h < \frac{1}{2}$. So we need examining case (i) and (ii) only as under.

Case (i) : Suppose $\max \{\|x_{k-1} - x_{k+1}, u_1, ..., u_{n-1}\|, \|x_{k-1} - x_k, u_1, ..., u_{n-1}\|, \|x_k - x_{k+1}, u_1, ..., u_{n-1}\|\}$ $= \|x_{k-1} - x_k, u_1, ..., u_{n-1}\|$

Therefore $\|x_k - x_{k+1}, u_1, ..., u_{n-1}\| \leq h\|x_{k-1} - x_k, u_1, ..., u_{n-1}\|$ (3.2)

Case (ii) : Suppose $\max \{\|x_{k-1} - x_{k+1}, u_1, ..., u_{n-1}\|, \|x_{k-1} - x_k, u_1, ..., u_{n-1}\|, \|x_k - x_{k+1}, u_1, ..., u_{n-1}\|\}$

$$= \|x_{k-1} - x_{k+1}, u_1, ..., u_{n-1}\|$$

Therefore $\|x_k - x_{k+1}, u_1, ..., u_{n-1}\| \leq h.\|x_{k-1} - x_{k+1}, u_1, ..., u_{n-1}\|$

$\leq h.\|x_{k-1} - x_k, u_1, ..., u_{n-1}\| + h.\|x_k - x_{k+1}, u_1, ..., u_{n-1}\|$ implies

$(1-h)\|x_k - x_{k+1}, u_1, ..., u_{n-1}\| \leq h\|x_{k-1} - x_k, u_1, ..., u_{n-1}\|$ (3.3)

From (3.2) and (3.3) we have

$\|x_k - x_{k+1}, u_1, ..., u_{n-1}\| \leq \max\left(h, \frac{h}{1-h}\right)\|x_{k-1} - x_k, u_1, ..., u_{n-1}\|$ for all *k* and for all $u_1, ..., u_{n-1} \in X$.

So, $\|x_k - x_{k+1}, u_1, ..., u_{n-1}\| \leq \left(\frac{h}{1-h}\right).\|x_{k-1} - x_k, u_1, ..., u_{n-1}\|$ for all *k* and for all $u_1, ..., u_{n-1} \in X$.

Proceeding in this way

$$\|x_k - x_{k+1}, u_1, ..., u_{n-1}\| \leq \left(\frac{h}{1-h}\right)^2 \|x_{k-2} - x_{k-1}, u_1, ..., u_{n-1}\|$$

$$\leq ... \leq \left(\frac{h}{1-h}\right)^k \|x_0 - x_1, u_1, ..., u_{n-1}\| = r^k \|x_0 - x_1, u_1, ..., u_{n-1}\|, \text{ where } r = \frac{h}{1-h} < 1.$$

If $s > k$, for $u_1, ..., u_{n-1} \in X$, we have

$$\|x_k - x_s, u_1, ..., u_{n-1}\| \leq \|x_k - x_{k+1}, u_1, ..., u_{n-1}\| + \|x_{k+1} - x_{k+2}, u_1, ..., u_{n-1}\| + ... + \|x_{s-1} - x_s, u_1, ..., u_{n-1}\|$$

$$\leq r^k (1 + r + r^2 + ... + r^{s-k-1}) \|x_0 - x_1, u_1, ..., u_{n-1}\| < \frac{r^k}{1-r}\|x_0 - x_1, u_1, ..., u_{n-1}\| \to 0 \text{ as } k \to \infty.$$

That means $\{x_k\}$ is Cauchy sequence in *X* and let $\lim_{k\to\infty} x_k = z \in X$.

Again for $u_1, ..., u_{n-1} \in X$, $\|x_{k+1} - T(z), u_1, ..., u_{n-1}\| = \|T(x_k) - T(z), u_1, ..., u_{n-1}\|$

$\leq h.\max \{\|x_k - z, u_1, ..., u_{n-1}\|, \|x_k - x_{k+1}, u_1, ..., u_{n-1}\|, \|z - T(z), u_1, ..., u_{n-1}\|,$

$\|x_k - T(z), u_1, ..., u_{n-1}\|, \|z - T(x_k), u_1, ..., u_{n-1}\|\}$

Passing on $\lim_{k\to\infty}$ we have $\|z - T(z), u_1, ..., u_{n-1}\| \leq h.\|z - T(z), u_1, ..., u_{n-1}\|$.

Therefore $z = T(z)$ and *z* is a fixed point of *T* where $z = \lim_{k\to\infty} T^k(x_0)$ for each $x_0 \in X$.

Uniqueness of *z* is obvious.

Theorem 3.2: Let *X* be a *n*-Banach space with $T : X \to X$. Let $\{T_k : X \to X\}$ be a sequence of mappings such that

(i) $\|T_k(x)-T_k(y),u_1,...,u_{n-1}\| \le h.\max\{\|x-y,u_1,...,u_{n-1}\|,\|x-T_k(x),u_1,...,u_{n-1}\|,$

$\|y-T_k(y),u_1,...,u_{n-1}\|,\|x-T_k(y),u_1,...,u_{n-1}\|,\|y-T_k(x),u_1,...,u_{n-1}\|\}$ for all $u_1,...,u_{n-1}$ and

$x,y\in X$, $k=1,2,...$ with $0<h<\frac{1}{2}$. (3.4)

and (ii) $\lim_{k\to\infty} T_k(x)=T(x)$ for each $x\in X$. Then T has a unique fixed point z in X such that $z=\lim_{k\to\infty} z_k$, z_k being the unique fixed point of T_k $(k=1,2,...)$.

Proof: Taking limit as $k\to\infty$ in (3.4) we obtain

$\|T(x)-T(y),u_1,...,u_{n-1}\| \le h.\max\{\|x-y,u_1,...,u_{n-1}\|,\|x-T(x),u_1,...,u_{n-1}\|,$

$\|y-T(y),u_1,...,u_{n-1}\|,\|x-T(y),u_1,...,u_{n-1}\|,\|y-T(x),u_1,...,u_{n-1}\|\}$ for all $u_1,...,u_{n-1}\in X$ and hence T satisfies (3.4), Hence by Theorem 3.1, T has a unique fixed point say $z\in X$.

Now for $u_1,u_2,...,u_{n-1}\in X$,

$$\|z-z_k,u_1,...,u_{n-1}\| = \|T(z)-T_k(z_k),u_1,...,u_{n-1}\|$$

$$\le \|T(z)-T_k(z),u_1,...,u_{n-1}\| + \|T_k(z)-T_k(z_k),u_1,...,u_{n-1}\| \quad (3.5)$$

Again $\|T_k(z)-T_k(z_k),u_1,...,u_{n-1}\|$

$\le h.\max\{\|z-z_k,u_1,...,u_{n-1}\|,\|z-T_k(z),u_1,...,u_{n-1}\|,\|z_k-T_k(z_k),u_1,...,u_{n-1}\|,$

$\|z_k-T_k(z),u_1,...,u_{n-1}\|,\|z-T_k(z_k),u_1,...,u_{n-1}\|\}$ implies

$\|T_k(z)-T_k(z_k),u_1,...,u_{n-1}\| \le h.\max\{\|z-z_k,u_1,...,u_{n-1}\|,\|z-T_k(z),u_1,...,u_{n-1}\|,\|z_k-z_k,u_1,...,u_{n-1}\|,$

$\|z_k-T_k(z),u_1,...,u_{n-1}\|,\|z-z_k,u_1,...,u_{n-1}\|\}$

$$= h\max\{\|z-z_k,u_1,...,u_{n-1}\|,\|z-T_k(z),u_1,...,u_{n-1}\|,\|z_k-T_k(z),u_1,...,u_{n-1}\|\} \quad (3.6)$$

By (3.5) and (3.6) we have,

$$\|z-z_k,u_1,...,u_{n-1}\| \le \|T(z)-T_k(z),u_1,...,u_{n-1}\|$$

$$+h.\max\{\|z-z_k,u_1,...,u_{n-1}\|,\|z-T_k(z),u_1,...,u_{n-1}\|,\|z_k-T_k(z),u_1,...,u_{n-1}\|\}$$

$$\le \|T(z)-T_k(z),u_1,...,u_{n-1}\| + h.(\|z-z_k,u_1,...,u_{n-1}\| + \|z-T_k(z),u_1,...,u_{n-1}\| + \|z_k-T_k(z),u_1,...,u_{n-1}\|)$$

or, $(1-h).\|z-z_k,u_1,...,u_{n-1}\| \le \|z-T_k(z),u_1,...,u_{n-1}\|$

$$+h.\|z-T_k(z),u_1,...,u_{n-1}\| + h.\|z_k-T_k(z),u_1,...,u_{n-1}\|$$

$$= (1+h)\|z-T_k(z),u_1,...,u_{n-1}\| + h.\|z_k-T_k(z),u_1,...,u_{n-1}\|$$

or, $\|z-z_k,u_1,...,u_{n-1}\| \le \frac{1+h}{1-h}\|z-T_k(z),u_1,...,u_{n-1}\| + \frac{h}{1-h}\|z_k-T_k(z),u_1,...,u_{n-1}\|$.

So, by routine calculation we get $z=\lim_{k\to\infty} z_k$.

Theorem 3.3: Let X be a n-Banach space and $T_k : X \to X$ be a sequence of mappings with fixed point z_k such that $T_k \;\to T$ uniformly over z_k to satisfy,

$$\|T\ x\ -T\ y\ ,u_1,...,u_{n-1}\| \le h.\max\ \|x-y,u_1,...,u_{n-1}\|,\|x-T\ x\ ,u_1,...,u_{n-1}\|,$$

$$\|y-T\ y\ ,u_1,...,u_{n-1}\|,\|x-T\ y\ ,u_1,...,u_{n-1}\|,\|y-T\ x\ ,u_1,...,u_{n-1}\| \qquad (3.7)$$

for all $x, y, u_1,...,u_{n-1} \in X$ where $0 < h < \frac{1}{2}$

then $\lim_{k\to\infty} z_k = z$ where z is the fixed point of T in X.

Proof: Fix $\varepsilon > 0$, from uniform convergence of T_k on $z_k : k = 1, 2, ...$ there exists an integer k such that for all $k \ge K$ and for all $u_1,...,u_{n-1} \in X$,

$$\|T\ z_k\ -T_k\ z_k\ ,u_1,...,u_{n-1}\| < \frac{\varepsilon}{M} \text{ for all } z_k \text{ where } M = \frac{1+h}{1-h}. \qquad (3.8)$$

Now $\|z - z_k, u_1,...,u_{n-1}\| = \|T\ z\ -T_k\ z_k\ ,u_1,...,u_{n-1}\|$

$$\le \|T\ z\ -T\ z_k\ ,u_1,...,u_{n-1}\| + \|T\ z_k\ -T_k\ z_k\ ,u_1,...,u_{n-1}\| \qquad (3.9)$$

From (3.7) we get,

$$\|T\ z\ -T\ z_k\ ,u_1,...,u_{n-1}\|$$

$$\le h.\max\ \|z-z_k,u_1,...,u_{n-1}\|,\|z-T\ z\ ,u_1,...,u_{n-1}\|,\|z_k-T\ z_k\ ,u_1,...,u_{n-1}\|,\|z-T\ z_k\ ,u_1,...,u_{n-1}\|,$$

$$\|z_k - T\ z\ ,u_1,...,u_{n-1}\|$$

$$\le h.\max\ \|z-z_k,u_1,...,u_{n-1}\|,\|z_k-T\ z_k\ ,u_1,...,u_{n-1}\|,\|z-T\ z_k\ ,u_1,...,u_{n-1}\|, \text{ since } z = T\ z\ .$$

Now $\|T\ z\ -T\ z_k\ ,u_1,...,u_{n-1}\| \le h.\|z-T\ z_k\ ,u_1,...,u_{n-1}\|$ is impossible because $z = T\ z$ and $0 < h < \frac{1}{2}$.

Hence above gives

$$\|T\ z\ -T\ z_k\ ,u_1,...,u_{n-1}\| \le h.\max\ \|z-z_k,u_1,...,u_{n-1}\|,\|z_k-T\ z_k\ ,u_1,...,u_{n-1}\|$$

Using (3.9) we have

$$\|z-z_k,u_1,...,u_{n-1}\| \le \|T\ z_k\ -T_k\ z_k\ ,u_1,...,u_{n-1}\|$$

$$+h.\max\ \|z-z_k,u_1,...,u_{n-1}\|,\|z_k-T\ z_k\ ,u_1,...,u_{n-1}\|$$

$$\le \|T\ z_k\ -T_k\ z_k\ ,u_1,...,u_{n-1}\| + h.\|z-z_k,u_1,...,u_{n-1}\|$$

$$+h.\|z_k-T\ z_k\ ,u_1,...,u_{n-1}\| \text{ implies}$$

$$1-h\ .\|z-z_k,u_1,...,u_{n-1}\| \le\ 1+h\ .\|z_k-T\ z_k\ ,u_1,...,u_{n-1}\|$$

i.e., $$\|z-z_k,u_1,...,u_{n-1}\| \le \left(\frac{1+h}{1-h}\right).\|T_k\ z_k\ -T\ z_k\ ,u_1,...,u_{n-1}\|$$

which equals to $M\|T_k\ z_k\ -T\ z_k\ ,u_1,...,u_{n-1}\| < \varepsilon$, for $k \ge N$ using (3.8). Hence $\lim_{k\to\infty} z_k = z \in X$.

Theorem 3.4: Let S and T be two self mappings of X satisfying

$$\|S\ \ x\ -T\ \ y\ \ ,u_1,...,u_{n-1}\| \le h\max\ \ \|x-y,u_1,...,u_{n-1}\|,\|x-S\ \ x\ \ ,u_1,...,u_{n-1}\|,$$
$$\|y-T\ \ y\ \ ,u_1,...,u_{n-1}\|,\frac{\|x-T\ \ y\ \ ,u_1,...,u_{n-1}\|+\|y-S\ \ x\ \ ,u_1,...,u_{n-1}\|}{2}\Bigg\} \qquad (3.10)$$

for all $x,y,u_1,...,u_{n-1}\in X$ where $0<h<\dfrac{1}{2}$. Then S and T have a unique common fixed point $z\in X$ where $\lim\limits_{n\to\infty} TS\ \ ^n\ \ x_0\ \ =z$ for each $x_0\in X$.

Proof: Let $x_0\in X$ and define $\ x_k\ $ by $x_{2k+1}=S\ \ x_{2k}$ and $x_{2k+2}=T\ \ x_{2k+1}$.

Then by (3.10)

$$\|x_{2k+1}-x_{2k+2},u_1,...,u_{n-1}\| = \|S\ \ x_{2k}\ \ -T\ \ x_{2k+1}\ \ ,u_1,...,u_{n-1}\|$$
$$\le h\max\ \ \|x_{2k}-x_{2k+1},u_1,...,u_{n-1}\|,\|x_{2k}-x_{2k+1},u_1,...,u_{n-1}\|,$$
$$\|x_{2k+1}-x_{2k+2},u_1,...,u_{n-1}\|,\frac{\|x_{2k}-x_{2k+2},u_1,...,u_{n-1}\|+\|x_{2k+1}-x_{2k+1},u_1,...,u_{n-1}\|}{2}\Bigg\}$$
$$= h\max\ \ \|x_{2k}-x_{2k+1},u_1,...,u_{n-1}\|,\|x_{2k+1}-x_{2k+2},u_1,...,u_{n-1}\|,$$
$$\frac{\|x_{2k}-x_{2k+2},u_1,...,u_{n-1}\|}{2}\Bigg\} \qquad (3.11)$$

Let $\dfrac{\|x_{2k}-x_{2k+2},u_1,...,u_{n-1}\|}{2}\ge\max\ \ \|x_{2k}-x_{2k+1},u_1,...,u_{n-1}\|,\|x_{2k+1}-x_{2k+2},u_1,...,u_{n-1}\|$ (3.12)

In case $\|x_{2k}-x_{2k+2},u_1,...,u_{n-1}\|\ge 2\|x_{2k}-x_{2k+1},u_1,...,u_{n-1}\|$

Then $\|x_{2k}-x_{2k+1},u_1,...,u_{n-1}\|+\|x_{2k+1}-x_{2k+2},u_1,...,u_{n-1}\|$

$\ge\|x_{2k}-x_{2k+2},u_1,...,u_{n-1}\|$

$\ge 2\|x_{2k}-x_{2k+1},u_1,...,u_{n-1}\|$, gives $\|x_{2k+1}-x_{2k+2},u_1,...,u_{n-1}\|\ge\|x_{2k}-x_{2k+1},u_1,...,u_{n-1}\|$

that means right hand side of (3.11) equals to $\|x_{2k+1}-x_{2k+2},u_1,...,u_{n-1}\|$, contrary to the case as assumed above.

Thus $\|x_{2k}-x_{2k+2},u_1,...,u_{n-1}\|\ngeq 2\|x_{2k}-x_{2k+1},u_1,...,u_{n-1}\|$

By a similar argument we arrive at

$\|x_{2k}-x_{2k+2},u_1,...,u_{n-1}\|\ngeq 2\|x_{2k+1}-x_{2k+2},u_1,...,u_{n-1}\|$

Therefore (3.11) is not tenable.

Further $\|x_{2k+1}-x_{2k+2},u_1,...,u_{n-1}\|\nleq h.\|x_{2k+1}-x_{2k+2},u_1,...,u_{n-1}\|$ and therefore (3.11) gives

$\|x_{2k+1}-x_{2k+2},u_1,...,u_{n-1}\|\le h.\|x_{2k}-x_{2k+1},u_1,...,u_{n-1}\|$

By exactly the same argument we produce

$\|x_{2k}-x_{2k+1},u_1,...,u_{n-1}\|\le h.\|x_{2k-1}-x_{2k},u_1,...,u_{n-1}\|$

for all $u_1,...,u_{n-1}\in X$ and therefore for all k we have $\|x_k-x_{k+1},u_1,...,u_{n-1}\|\le h.\|x_{k-1}-x_k,u_1,...,u_{n-1}\|$

$\le h^2.\|x_{k-2}-x_{k-1},u_1,...,u_{n-1}\|$

$\le ...\le h^k.\|x_0-x_1,u_1,...,u_{n-1}\|$.

NOW following standard and usual arguments one reaches to conclude that $\{x_k\}$ is a Cauchy in X, and let $\lim_{k\to\infty} x_k = z \in X$. Now for $u_1, ..., u_{n-1} \in X$,

$$\|x_{2k+2} - S(z), u_1, ..., u_{n-1}\| = \|T(x_{2k+1}) - S(z), u_1, ..., u_{n-1}\|$$
$$\leq h.\max\Big[\|x_{2k+1} - z, u_1, ..., u_{n-1}\|,$$
$$\|x_{2k+1} - x_{2k+2}, u_1, ..., u_{n-1}\|, \|z - S(z), u_1, ..., u_{n-1}\|,$$
$$\frac{1}{2}\left(\|x_{2k+1} - S(z), u_1, ..., u_{n-1}\| + \|z - x_{2k+2}, u_1, ..., u_{n-1}\|\right)\Big]$$

Passing on $\lim_{k\to\infty}$ above gives

$$\|z - S(z), u_1, ..., u_{n-1}\| \leq h\|z - S(z), u_1, ..., u_{n-1}\|$$

and we conclude that $z = S(z)$.

Similarly, we show that $z = T(z)$ and hence z is a common fixed point of S and T.

Uniqueness of z is obvious by virtue of (3.10). Finally taking $x = x_0 \in X$, and following iteration scheme as undertaken we have $(TS)^k(x_0) = x_{2k}$. So,

$$\lim_{k\to\infty}(TS)^k(x_0) = \lim_{k\to\infty} x_{2k} = z \in X.$$

The proof is now complete.

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