Annular Bounds for the Zeros of Complex Polynomials

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Research Article

Abstract: The location of zeros of complex polynomials has been investigated in frame work of Enestrom and Kakeya theorem. In this paper we extend some existing results on the zeros of complex polynomials by considering restrictions on its coefficients.

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Introduction

The following result due to Enestrom and Kakeya [12] is well known in the theory of distribution of zeros of polynomials.

Theorem A (1): If P (z) $=\sum_0^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \text{------} \geq a_1 \geq a_0 > 0 \text{ , } a_j \in R \tag{1}$$

Then P (z) does not vanish in $|z|>1$
This is a very elegant result but it is equally limited in scope as the hypothesis is very restrictive.
A. Joyal *et al* [11] extended this theorem to the polynomials whose coefficient are monotonic but not necessarily non negative and proved the following:

Theorem A (2): If P (z) $=\sum_0^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \text{------} \geq a_1 \geq a_0 \text{ , } a_j \in R$$

Then all the zeros of P (z) lie in

$$|z| \leq (a_n - a_0 + |a_0|) \div |a_n|. \tag{2}$$

This was further improved upon by Dewan and Govil[7].
Shah and Liman [15] relaxed the hypothesis and proved the following result.

Theorem B: Let P (z) $=\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. If
Re $(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j$, for $j = 0,1,2\text{-----}n$. such that for some $\lambda \geq 1$,
$\lambda\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \text{------} \geq \alpha_1 \geq \alpha_0$,
$\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \text{------} \geq \beta_1 \geq \beta_0 > 0$
Then all the zeroes of P (z) lie in

$$|z + \frac{\alpha_n}{a_n}(\lambda -1)| \leq [\lambda\alpha_n - \alpha_0 + |\alpha_0| + \beta_n] \div |a_n| \tag{3}$$

Aziz and Zargar [1] relaxed the hypothesis of Theorem A (1) and proved the following extensions of Enestrom-Kakeya theorem.

Theorem C: Let P (z) $=\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients such that for some $k \geq 1$,
$ka_n \geq a_{n-1} \geq \text{-----} a_1 \geq a_0 > 0$
Then all the zeros of P(z) lie in $|z+k-1| \leq k$ (4)

Theorem D: Let P (z) $=\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. If
Re $(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j$, for $j = 0,1,2\text{-----}n$. such that for some $k \geq 1$,
$\lambda\alpha_n \leq \alpha_{n-1} \leq \text{----} \leq \alpha_{p+1} \leq \alpha_p \geq \alpha_{p-1} \geq \text{----} \geq \alpha_1 \geq \alpha_0$
$\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \text{------} \geq \beta_1 \geq \beta_0 > 0$
Where $0 \leq p \leq n-1$, then all the zeros of P(z) lie in

$$|z + \frac{\alpha_n}{a_n}(\lambda -1)| \leq [2\alpha_p - \lambda\alpha_n - \alpha_0 + |\alpha_0| + \beta_n] \div |a_n| \tag{5}$$

Recently, Choo [5] has proved the following theorem

Theorem E: Let P (z) $=\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. If Re $(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for j = 0,1,2-----n. such that for some p and r and for some λ, $\mu > 0$

$\lambda\alpha_n \le \alpha_{n-1} \le ----\le \alpha_{p+1} \le \alpha_p \ge \alpha_{p-1} \ge ---- \ge \alpha_1 \ge \alpha_0$

$\mu\beta_n \le \beta_{n-1} \le ---- \le \beta_{r+1} \le \beta_r \ge \beta_{r-1} \ge ---- \ge \beta_1 \ge \beta_0$

Then P(z) has all its zeros in $R_1 \le |z| \le R_2$ where

$R_1 = \frac{|\alpha_0|}{M_1}$ and $R_2 = \frac{M_2}{|\alpha_n|}$

With

$M_1 = |\alpha_n| + |(\lambda-1)\alpha_n| + |(\mu-1)\beta_n| + 2(\alpha_p+\beta_r) - (\lambda\alpha_n + \mu\beta_n) - (\alpha_0+\beta_0)$

And

$M_2 = |(\lambda-1)\alpha_n| + |(\mu-1)\beta_n| + 2(\alpha_p+\beta_r) - (\lambda\alpha_n + \mu\beta_n) - (\alpha_0+\beta_0) + |\alpha_0|$

Here we notice that the annulus $R_1 \le |z| \le R_2$ is expressed in terms of λ and μ as associated to the coefficients α_n and β_n in the given constraint in Theorem E.

Theorem 1: Let P (z) $=\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. If Re $(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for j = 0,1,2-----n. such that for some $\delta,\eta \ge 1$ and $\tau,\sigma \le 1$

$\delta\alpha_n \le \alpha_{n-1} \le ----\le \alpha_{p+1} \le \alpha_p \ge \alpha_{p-1} \ge ---- \ge \alpha_1 \ge \tau\alpha_0$

$\eta\beta_n \le \beta_{n-1} \le ---- \le \beta_{q+1} \le \beta_q \ge \beta_{q-1} \ge ---- \ge \beta_1 \ge \sigma\beta_0$ (6)

where $0 \le p,q \le n-1$, then all the zeros of P(z) lie in the disk

$R^{\delta\eta} \le |z - z_{\delta\eta}| \le R_{\delta\eta}$, (7)

where

$z_{\delta\eta} = -[\frac{(\delta-1)\alpha_n}{a_n} + i\frac{(\eta-1)\beta_n}{a_n}]$, (8a)

$R_{\delta\eta} = \frac{1}{|a_n|}[2(\alpha_p+\beta_q) - (\delta\alpha_n + \eta\beta_n) - \tau\alpha_0 + (1-\tau)\alpha_0 - \sigma\beta_0 + (1-\sigma)\beta_0 + |\alpha_0|$ (8b)

$R^{\delta\eta} = \frac{|a0|}{|a_n| + (\delta-1)|\alpha_n + (\eta-1)|\beta_n| + 2(\alpha_p + \beta q) - (\delta\alpha_n + \eta\beta_n) - \tau\alpha_0 + (1-\tau)\alpha_0 - \sigma\beta 0 + (1-\sigma)\beta 0}$

$-\frac{1}{|a_n|}[(\delta-1)^2\alpha_n^2 + (\eta-1)^2\beta_n^2]^{1/2}$ (8c)

Proof: Consider the polynomial

F (z) = (1-z) P (z)

$= -z^n\{(\alpha_n + i\beta_n)z + (\delta-1)\alpha_n + i(\eta-1)\beta_n\} + [(\delta\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + --------------------- + \{(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)z + \alpha_0] + i[(\eta\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + ------ + \{(\beta_1 - \sigma\beta_0) + (\sigma\beta_0 - \beta_0)\}z + \beta_0]$

Now if $|z|>1$, $\frac{1}{|z|^{n-j}} < 1$, j= 0, 1, 2---n-1

Therefore,

$|F(z)| \ge |z|^n [|a_n z + (\delta-1)\alpha_n + i(\eta-1)\beta_n| - \{2\alpha_p + 2\beta_q - \delta\alpha_n - \eta\beta n - \tau\alpha_0 + (\alpha_0)(1-\tau) - \sigma\beta_0 + (\beta_0)(1-\sigma) + |\alpha_0|\}]$

>0, if

$|z + \frac{(\delta-1)\alpha_n}{a_n} + i\frac{(\eta-1)\beta_n}{a_n}| >$

$\frac{1}{|a_n|}\{2(\alpha_p + \beta_q) - (\delta\alpha_n + \eta\beta n) - \tau\alpha_0 + (\alpha_0)(1-\tau) - \sigma\beta_0 + (\beta_0)(1-\sigma) + |\alpha_0|\}$

This shows that the zeros of F (z) having modulus greater than 1 lie in

$\{|z + \frac{(\delta-1)\alpha_n}{a_n} + i\frac{(\eta-1)\beta_n}{a_n}| \le$

$\frac{1}{|a_n|}\{2(\alpha_p + \beta_q) - (\delta\alpha_n + \eta\beta n) - \tau\alpha_0 + (\alpha_0)(1-\tau) - \sigma\beta_0 + (\beta_0)(1-\sigma) + |\alpha_0|\}$ (9)

Since all the zeros of P(z) with modulus greater than 1 lie in the disc given by eq(9), it can be shown that $R_{\delta\eta} \ge 1$. Consequently the zeros of P (z) with modulus less than or equal to one are already contained in the disk $|z - z_{\delta\eta}| \le R_{\delta\eta}$. (10)

In order to prove the lower bound $R_{\delta\eta} \leq |z-z_{\delta\eta}|$ we first prove the following lemma.

Lemma: Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. Then for |z|<1, we show that

$$|z| \leq \frac{|a0|}{M_2} = \frac{|a0|}{|an|+(\delta-1)|\alpha_n|+(\eta-1)|\beta_n|+2(\alpha p+\beta q)-(\delta\alpha n+\eta\beta n)-\tau\alpha_0+(\alpha_0)(1-\tau)-\sigma\beta_0+(\beta_0)(1-\sigma)}$$

Proof: Let |z|<1.

Consider F (z) = (1-z) P (z)

$$= \chi(z) + a_0, \quad (11)$$

Where

$\chi(z) = \{(\alpha_n + i\beta_n)z + (\delta-1)\alpha_n + i(\eta-1)\beta_n\} + [(\delta\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \text{------------} + \{(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)z + \quad i[(\eta\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \text{------} + \{(\beta_1 - \sigma\beta_0) + (\sigma\beta_0 - \beta_0)\}z]$

$\therefore |\chi(z)| = |\{(\alpha_n + i\beta_n)z + (\delta-1)\alpha_n + i(\eta-1)\beta_n\} + [(\delta\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \text{---------} +$
$\{(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)z + i[(\eta\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \text{------} + \{(\beta_1 - \sigma\beta_0) + (\sigma\beta_0 - \beta_0)\}z]|$
$\leq |a_n z| + |(\delta-1)\alpha_n| + |(\eta-1)\beta_n| + [M_1]$

Where

$$M_1 = 2(\alpha_p + \beta_q) - (\delta\alpha_n + \eta\beta n) - \tau\alpha_0 + (\alpha_0)(1-\tau) - \sigma\beta_0 + (\beta_0)(1-\sigma) \quad (12)$$

Since $\chi(0)=0$, it follows by Schwarz lemma that

$|\chi(z)| \leq M_1|z|$ for |z|<1

Therefore for |z|<1,

$|F(z)| = |\chi(z) + a_0| \geq |a_0| - |\chi(z)| > 0$, if

$|a_0| > |z|[M_2]$,

where $M_2 = |a_n| + (\delta-1)|\alpha_n| + (\eta-1)|\beta_n| + M_1$

$$= |a_n| + (\delta-1)|\alpha_n| + (\eta-1)|\beta_n| + 2(\alpha_p + \beta_q) - (\delta\alpha_n + \eta\beta n) - \tau\alpha_0 + (\alpha_0)(1-\tau) - \sigma\beta_0 + (\beta_0)(1-\sigma) \quad (13)$$

Thus, $|z| \leq \frac{|a0|}{M_2}$

$$= \frac{|a0|}{|an|+(\delta-1)|\alpha_n|+(\eta-1)|\beta_n|+2(\alpha p+\beta q)-(\delta\alpha n+\eta\beta n)-\tau\alpha_0+(\alpha_0)(1-\tau)-\sigma\beta_0+(\beta_0)(1-\sigma)} \quad (14)$$

Hence P(z) does not vanish in $|z| < \frac{|a0|}{M_2}$. It can be shown that $M_2 \leq |a_0|$ so that $|z| \leq 1$. Hence P (z) has all its zeros in $\frac{|a0|}{M_2} \leq |z|$. (15)

Now we prove the second part of the main theorem (1)

Since $|z - z_{\delta\eta}| \geq |z| - |z_{\delta\eta}|$, (16)

then using eq(15) of above lemma in eq(16), we have

$|z - z_{\delta\eta}| \geq |z| - |z_{\delta\eta}| \geq \frac{|a0|}{M_2} - |z_{\delta\eta}|$

This implies $\frac{|a_0|}{M_2} - |z_{\delta\eta}| \leq |z - z_{\delta\eta}|$

$$\frac{|a0|}{M_2} - \left| \frac{(\delta-1)\alpha_n}{a_n} + i\frac{(\eta-1)\beta_n}{a_n} \right| \leq |z - z_{\delta\eta}| \quad (17)$$

From eq(17) we obtain $R^{\delta\eta} \leq |z-z_{\delta\eta}|$, (18)

where $R^{\delta\eta}$ is given in eq 8(c)

On combining eq(10) and eq(18) the above theorem is completely proved.

Conclusion

We get (i) if $\tau = 1$, $\sigma \neq 1$, then all the zeros of P(z) lie in the disk

$R^{22} \leq |z-z_{\delta\eta}| \leq R_{11}$, (19)

where,

$R_{11} = \frac{1}{|a_n|}[2(\alpha_p + \beta_q) - (\delta\alpha_n + \eta\beta_n) - \alpha_0 - \sigma\beta_0 + (1-\sigma)\beta_0 + |a_0|$ (19a)

$$R^{22} = \frac{|a0|}{|a_n|+(\delta-1)|\alpha_n+(\eta-1)|\beta_n|+2(\alpha_p+\beta q)-(\delta\alpha_n+\eta\beta_n)-\alpha_0-\sigma\beta 0+(1-\sigma)\beta 0}$$

$-\frac{1}{|a_n|}[(\delta-1)^2\alpha_n{}^2+(\eta-1)^2\beta_n{}^2]^{1/2}$ (19b)

and

(ii) if $\sigma = 1$, $\tau \neq 1$, then all the zeros of P(z) lie in the disk

$R^{44} \leq |z-z_{\delta\eta}| \leq R_{33}$, (20)

where ,

$R_{33} = \frac{1}{|a_n|}[2(\alpha_p+\beta_q)-(\delta\alpha_n+\eta\beta_n)-\alpha_0-\sigma\beta_0+(1-\sigma)\beta_0+|a_0|$ (20a)

$R^{44} = \frac{|a0|}{|a_n|+(\delta-1)|\alpha_n+(\eta-1)|\beta_n|+2(\alpha_p+\beta q)-(\delta\alpha_n+\eta\beta_n)-\alpha_0-\sigma\beta 0+(1-\sigma)\beta 0}$

$-\frac{1}{|a_n|}[(\delta-1)^2\alpha_n{}^2+(\eta-1)^2\beta_n{}^2]^{1/2}$ (20b)

and $z_{\delta\eta}$ is given by eq(8a)

(iii) Further we note with regard to the upper bound of the Theorem 1 given as $|z-z_{\delta\eta}| \leq R_{\delta\eta}$,

where

$z_{\delta\eta} = -[\frac{(\delta-1)\alpha_n}{a_n}+i\frac{(\eta-1)\beta_n}{a_n}] = A+iB$ where $A = -\frac{(\delta-1)\alpha_n}{a_n}$ and $B = -\frac{(\eta-1)\beta_n}{a_n}$

and

$R_{\delta\eta} = \frac{1}{|a_n|}[2(\alpha_p+\beta_q)-(\delta\alpha_n+\eta\beta_n)-\tau\alpha_0+(1-\tau)\alpha_0-\sigma\beta_0+(1-\sigma)\beta_0+|a_0|]$

and that if we transfer the centre of the above disc at the origin so that equation (9) can be written as

$|z| = |\overline{z-z_{\delta\eta}}+z_{\delta\eta}| \leq |z-z_{\delta\eta}|+|z_{\delta\eta}|$

$\leq R_{\delta\eta}+|z_{\delta\eta}|$

$\leq \frac{1}{|a_n|}\{2(\alpha_p+\beta_q)-(\delta\alpha_n+\eta\beta n)-\tau\alpha_0+(\alpha_0)(1-\tau)-\sigma\beta_0+(\beta_0)(1-\sigma)+|a_0|\}+\sqrt{A^2+B^2}$ (21)

Comparing this bound with upper bound of Theorem E given by:

$|z| \leq R_{11} = \frac{M_2}{|a_n|}$

$\leq \frac{1}{|a_n|}\{|(\lambda-1)\alpha_n|+|(\mu-1)\beta_n|+2(\alpha_p+\beta_r)-(\lambda\alpha_n+\mu\beta_n)-(\alpha_0+\beta_0)+|a_0|\}$

$\leq \frac{1}{|a_n|}[2(\alpha_p+\beta_q)-(\lambda\alpha_n+\mu\beta_n)-(\alpha_0+\beta_0)+|a_0|]+|A|+|B|$ (20)

We here find that the present bound given by (21) corresponding to $\tau = 1 = \sigma$ is sharper than eq (20) of Choo [5], in view of $\sqrt{A^2+B^2} < A+B$.

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