A Note on the Finite Sample Properties of Ridge Estimator

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Research Article

Abstract: The article studies the finite sample properties of generalized ridge regression estimator using a different approach. A comparative study of relative bias and relative efficiency of the estimator with respect to the ordinary least square have been made empirically. The results have also been compared with the existing results and are found to be quite different from those already existing in literature.

Keywords: Ordinary Least Squares, Generalized Ridge Regression (GRR), Relative Bias, Relative MSE, Relative Efficiency, Finite Sample Properties.

Introduction

In linear regression models, Ridge Regression is perhaps the most widely used technique in the presence of multicollinearity. Proposed by Hoerl and Kennard [3], [4] the Ridge Regression Estimator is characterized by a scalar, the choice of which is subjective requiring the judgment of the analyst. However, working with the canonical form of the regression model Hoerl and Kennard [3] defined general ridge regression estimator suggesting an initial choice of the characterizing scalar. Extensive work has been carried out since then, a good account of which is available in Vinod and Ullah [7] and [6]. Working with the initial choice of the characterizing scalar, Dwivedi *et al.* [1] worked out the first two moments of individual coefficients of GRR estimator assuming error distribution to be normal. Hemmerle and Carey[2] also worked out exact properties of two different forms of GRR estimators but demonstrated that the one suggested by Hoerl and Kennard [3] performs better in terms of relative bias and relative risk. It will not be out of context to mention that the estimators perform differently when sample size is small and it is more so in the presence of multicollinearity as negative effects of multicollinearity are magnified in smaller samples. Owing to this, assuming error distribution to be normal the paper attempts to assess the finite sample behavior of generalized ridge regression. For this purpose the relative bias, relative mean squared error and the relative efficiency of the estimator in comparison to OLS have been evaluated numerically and compared with the existing results. Interestingly, the expressions of relative bias and relative risk are found to be different from the existing results obtained by Dwivedi *et al.* [1]. The following section describes the estimator and its properties and empirically enumerates the results. A brief outline of proof of the theorem is provided in the end.

The Estimator and its Properties

Consider the canonical form of the linear regression model

$$y = X\beta + u \tag{2.1}$$

where y is an $n \times 1$ vector of n observation on the dependent variables, X is an $n \times p$ full column rank matrix of n observation of p explanatory variables, and β is a $p \times 1$ vector of unknown regression coefficients . The elements of disturbance vector u are assumed to be i.i.d each following normal distribution with mean zero and variance σ^2 so that

$$E[u] = 0 \text{ and } E[uu'] = \sigma^2 I_n \text{ ; } u \sim N(0, \sigma^2 I_n) \tag{2.2}$$

Following [3], we can write

$$X'X = \wedge \text{ where } \wedge = diag[\lambda_1, \lambda_2 \dots \lambda_p]$$

This canonical reduction can be obtained by using singular value decomposition of the $n \times p$ matrix X (see, [7], p5-6). Using it the general ridge regression estimator is given by

$$\beta^* = (\wedge + K)^{-1} X' y \tag{2.3}$$

$$= \Delta b \text{ where } \Delta = (\wedge + K)^{-1} \wedge$$

and K is a diagonal matrix with nonnegative elements $k_1, k_2, \dots k_p$ as the characterizing scalar, and

$$b = \wedge^{-1} X' y$$

is the ordinary least square estimator of β. Clearly, GRR is a biased estimator, with bias vector

$$Bias(\beta^*) = E(\beta^* - \beta) \tag{2.4}$$

$$= [\Delta - I_p]\beta$$

and

$$MSE(\beta^*) = \sigma^2\Delta^2 \wedge^{-1} + (\Delta - I_p)\beta\beta'(\Delta - I_p)$$
$$= \sigma^2\Delta \wedge^{-1} \Delta' \tag{2.5}$$

as $\wedge$ and K are assumed to be diagonal matrices, Δ is also diagonal and

$$tr\, MSE(\beta^*) = \sum_{i=1}^{p} \frac{\sigma^2\lambda_i + \beta_i^2 k_i^2}{(\lambda_i + k_i)^2} \tag{2.6}$$

Provided k_i 's are non-stochastic.

Now minimizing the expression (2.6) term by term i.e. minimizing the diagonal elements of the mean squared error matrix of (2.5) with respect to k_i yields

$$k_{i(opt)} = \frac{\sigma^2}{\beta_i^2} \ (i = 1,2,...,p) \tag{2.7}$$

[3] Suggested to start with

$$\frac{s^2}{b_i^2} = \hat{k}_i (say) \tag{2.8}$$

where b_i is the ith element of the ordinary least squares estimator b of β and

$$s^2 = \frac{1}{v}(y - Xb)'(y - Xb) \tag{2.9}$$

is an unbiased estimator of σ^2 where $v = (n - p)$

Using (2.8) in (2.3) leads to an adaptive estimator of β as

$$\hat{\beta} = (\wedge + \hat{k})^{-1}X'y \tag{2.10}$$

where the ith element of $\hat{\beta}$ is given by

$$\hat{\beta}_i = \frac{\lambda_i}{\lambda_i + \hat{k}_i} b_i \tag{2.11}$$

For finite sample sizes, the following theorem gives the first and second moments of $\hat{\beta}_i$

Theorem: Assuming normality of errors the first and second moments of $\hat{\beta}_i$ of (2.11) are given by

$$E[\hat{\beta}_i] = \beta_i e^{-\frac{\tau_i^2}{2}}\left[\sum_{j=1}^{\infty} \frac{\left(\frac{\tau_i^2}{2}\right)^{j-1} v^{-\left(j+\frac{1}{2}\right)}}{\Gamma\left(j+\frac{1}{2}\right)\Gamma(j)} \cdot \sum_{r=0}^{\infty}\left(\frac{v-1}{v}\right)^r \cdot \frac{\Gamma\left(j+\frac{v}{2}+\frac{1}{2}+r\right)}{\Gamma(r+1)} \cdot \frac{\Gamma\left(j+r+\frac{3}{2}\right)}{\Gamma\left(j+r+\frac{3}{2}+\frac{v}{2}\right)}\right] \tag{2.12}$$

$$E[\hat{\beta}_i]^2 = \beta_i^2 e^{-\frac{\tau_i^2}{2}}\left[\sum_{j=1}^{\infty} \frac{(2j-1)\left(\frac{\tau_i^2}{2}\right)^{j-2} v^{-\left(j+\frac{1}{2}\right)}}{2.\Gamma\left(j+\frac{1}{2}\right)\Gamma(j)} \sum_{r=0}^{\infty}\left(\frac{v-1}{v}\right)^r \cdot \frac{\Gamma\left(j+\frac{v}{2}+\frac{1}{2}+r\right)}{\Gamma(r+1)} \cdot \frac{\Gamma\left(j+r+\frac{5}{2}\right)}{\Gamma\left(j+r+\frac{5}{2}+\frac{v}{2}\right)}\right] \tag{2.13}$$

where $\tau_i^2 = \frac{\lambda_i \beta_i^2}{\sigma^2}$ is the non-centrality parameter. Using (2.12) and (2.13) we can compute the bias and mean squared error of $\hat{\beta}_i$.

Proof: see Appendix.

Using these, it is easy to compute the relative bias and relative mean squared error using

$$RB(\hat{\beta}_i) = E\left(\frac{\hat{\beta}_i - \beta_i}{\beta_i}\right) \tag{2.14}$$

and

$$RMSE(\hat{\beta}_i) = E\left(\frac{\hat{\beta}_i - \beta_i}{\beta_i}\right)^2 \tag{2.15}$$

Respectively. The efficiency of the OLS relative to GRR estimator is obtained from

$$\eta_i = \frac{MSE(\hat{\beta}_i)}{V(b_i)} \times 100 \tag{2.16}$$
$$= 100\tau_i^2 RMSE(\hat{\beta}_i)$$

The values of $RB(\hat{\beta}_i), RMSE(\hat{\beta}_i)$ and η_i have been tabulated for a few selected vales of $\frac{\tau_i^2}{2}$ and v .These results are provided in Tables 2.1, 2.2, 2.3 respectively and have been graphed for selected values of $\frac{\tau_i^2}{2}$ and v.

The expressions (2.12) and (2.13) are clearly different from those obtained by Dwivedi *et al.* [1] and therefore a substantial difference is observed numerically. Unlike the results obtained by Dwivedi *et al.* [1], the magnitude of relative bias is found to be decreasing function of the non-centrality parameter and an increasing function of degrees of freedom, so long as $\frac{\tau_i^2}{2} > 5$ and $v < 4$. However, for $v \geq 4$ the magnitude of relative bias is found to be increasing. Interestingly, as v increases the relative bias tends to -1, justification of this easily comes from the fact that the Ridge regression estimator shrinks the parameter vector towards zero. The relative MSE and relative efficiency are also observed to be decreasing for specific values of v and $\frac{\tau_i^2}{2}$. Hence, the finite sample properties of the Ridge regression estimator are not only heavily dependent upon the non-centrality parameter but on the degrees of freedom as well. It is also pertinent to mention that ambiguity in the numerical computations in relative bias, relative MSE and relative efficiency are found in the paper by Dwivedi *et al.* [1] when $\frac{\tau_i^2}{2} > 5$ which are evident in the respective tables.

Table 2.1: Relative Bias for specific values of non-centrality parameter and degrees of freedom

$\frac{\tau_i^2}{2} \downarrow$	$\rightarrow v$	1	2	5	10	20	50
0.01	E0	-0.24917	-0.28795	-0.46717	-0.73385	-0.9078	-0.98448
	E1	-0.249	-0.287	-0.318	-0.33	-0.337	-0.353
0.05	E0	-0.24588	-0.28419	-0.47043	-0.73901	-0.91041	-0.98502
	E1	-0.246	-0.283	-0.313	-0.325	-0.332	-0.348
0.1	E0	-0.24187	-0.27962	-0.47464	-0.74536	-0.91358	-0.98566
	E1	-0.242	-0.278	-0.307	-0.319	-0.326	-0.341
0.5	E0	-0.21306	-0.24776	-0.5126	-0.79201	-0.93533	-0.98992
	E1	-0.213	-0.243	-0.268	-0.278	-0.283	-0.296
0.7	E0	-0.2006	-0.2347	-0.53362	-0.81263	-0.94412	-0.99155
	E1	-0.201	-0.228	-0.251	-0.26	-0.265	-0.276
0.9	E0	-0.18924	-0.22337	-0.55544	-0.83154	-0.95175	-0.99292
	E1	-0.189	-0.215	-0.236	-0.244	-0.248	-0.258
1	E0	-0.18394	-0.21832	-0.56653	-0.84038	-0.95518	-0.99352
	E1	-0.184	-0.208	-0.228	-0.236	-0.241	-0.25
2	E0	-0.14196	-0.18716	-0.67665	-0.90889	-0.97873	-0.99732
	E1	-0.142	-0.158	-0.172	-0.177	-0.18	-0.185
5	E0	-0.1107	**-0.25493**	-0.89913	-0.98566	-0.99789	-0.99982
	E1	-0.08	-0.086	-0.091	-0.093	-0.094	-0.095
10	E0	**-0.56774**	**-0.70662**	**-0.99125**	**-0.99952**	**-0.99996**	**-1**
	E1	**-0.045**	**-0.047**	**-0.048**	**-0.049**	**-0.049**	**-0.049**
50	E0	-1	-1	-1	-1	-1	-1
	E1	-0.014	-0.014	-0.014	-0.014	-0.014	-0.014

E0 –Results by our approach
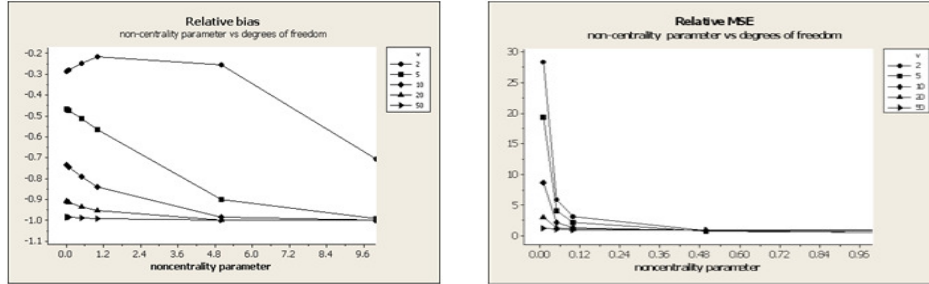E1-Results by Dwivedi *et al.* [1].

Table 2.2: Relative MSE for specific values of non-centrality parameter and degrees of freedom

$\frac{\tau_i^2}{2} \downarrow$	$\rightarrow v$	1	2	5	10	20	50
0.01	E0	31.53006	28.35765	**19.30435**	**8.598509**	**2.945015**	**1.17679**
	E1	31.53	28.423	**28.85**	**24.837**	**24.267**	**22.754**
0.05	E0	6.525343	5.922473	4.079349	2.146142	1.245423	1.010629
	E1	6.525	5.94	5.454	5.263	5.155	4.87
0.1	E0	3.394597	3.111778	2.178566	1.348879	1.03842	0.991026
	E1	3.395	3.123	2.898	2.809	2.758	2.626
0.5	E0	0.853875	0.819369	0.686614	0.779676	0.909809	0.982917
	E1	0.854	0.827	0.804	0.795	0.789	0.776
0.7	E0	0.657999	0.63841	0.598543	0.768053	0.914648	0.985032
	E1	0.658	0.646	0.635	0.631	0.628	0.622
0.9	E0	0.543213	0.530987	0.561328	0.774361	0.922711	0.987153
	E1	0.543	0.538	0.533	0.531	0.53	0.527
1	E0	0.501285	0.570042	0.552523	0.780415	0.927023	0.988142
	E1	0.501	0.498	0.495	0.494	0.493	0.492
2	E0	0.289568	0.292192	0.593021	0.858859	0.962787	0.994925

	E1	0.29	0.292	0.294	0.295	0.295	0.296
5	E0	0.118409	0.213709	0.854008	0.975889	0.996133	0.999642
	E1	0.123	0.123	0.124	0.124	0.124	0.124
10	E0	**0.455941**	**0.620464**	**0.986397**	**0.999155**	**0.999928**	**0.999996**
	E1	**0.059**	**0.058**	**0.058**	**0.058**	**0.057**	**0.057**
50	E0	1	1	1	1	1	1
	E1	0.013	0.013	0.013	0.013	0.013	0.013

E0 –Results by our approach
E1-Results by Dwivedi *et al*. [1].

Graph: Showing relative bias and relative MSE of GRR Estimator (Figure 1 and Figure2)

Table 2.3: Relative Efficiency for specific values of non-centrality parameter and degrees of freedom

$\frac{\tau_i^2}{2}$ ↓	$\to v$	1	2	5	10	20	50
0.01	E0	63.06011	56.7153	38.6087	17.19702	5.890029	2.35358
	E1	63.06	56.846	51.701	49.674	48.534	45.508
0.05	E0	65.25343	59.22473	40.79349	21.46142	12.45423	10.10629
	E1	65.253	59.398	54.542	52.626	51.548	48.698
0.1	E0	67.89194	62.23557	43.57132	26.97759	20.76841	19.82052
	E1	67.892	62.465	57.955	56.17	55.167	52.527
0.5	E0	85.3875	81.93687	68.66142	77.96756	90.98089	98.29169
	E1	85.388	82.69	80.394	79.46	78.937	77.632
0.7	E0	92.11985	89.37737	83.79599	**107.5274**	128.0507	137.9045
	E1	92.12	90.405	88.907	88.282	87.936	87.106
0.9	E0	97.77838	95.57758	**101.039**	139.385	166.088	177.6875
	E1	97.779	96.845	95.985	95.607	95.403	94.949
1	E0	**100.2571**	114.0084	110.5045	156.0829	185.4047	197.6283
	E1	**100.258**	99.65	99.056	98.783	98.638	98.34
2	E0	115.8271	116.8768	237.2084	343.5434	385.1149	397.9702
	E1	115.883	116.927	117.683	117.94	118.099	118.569
5	E0	118.4089	213.7089	854.0076	975.8885	996.1333	999.6423
	E1	123.245	123.494	123.602	123.635	123.653	123.833
10	E0	**911.8829**	**1240.929**	**1972.794**	**1998.311**	**1999.857**	**1999.992**
	E1	**117.495**	**116.319**	**115.394**	**115.026**	**114.891**	**114.656**
50	E0	10000	10000	10000	10000	10000	10000
	E1	129.538	128.682	128.197	128.039	127.988	127.827

E0 –Results by our approach
E1-Results by Dwivedi *et al*. [1].

Appendix

In order to find the expression (2.12) of the theorem, let us define

$$z_i = x_i'y/\sigma\sqrt{\lambda_i} = \lambda_i\,\beta_i/\sigma \tag{A.1}$$

where x_i (i=1,2…p) is the ith column vector of X. Since b_i is the OLS estimator of β_i following

$N(\beta_i, \frac{\sigma^2}{\lambda_i})$, therefore, the distribution of $z_i \sim N(\delta, 1)$ where $\delta = \tau_i$.

Next, the distribution of $v\frac{s^2}{\sigma^2}$ is χ^2 with v degrees of freedom and is independent of the distribution of z_i. Using z_i, we can write

$$\hat{\beta}_i = \frac{\sigma}{\sqrt{\lambda_i}} \frac{z_i^3}{z_i^2+c} \; ; c = \frac{s^2}{\sigma^2} \tag{A.2}$$

$$E[\hat{\beta}_i] = \frac{\sigma}{\sqrt{\lambda_i}} E[\emptyset(z_i^2,c) * z_i] \tag{A.3}$$

$$E[\emptyset(z_i^2,c) * z_i] = e^{-\frac{\delta^2}{2}} \int_{-\infty}^{\infty} \int_0^{\infty} \emptyset(z_i^2,c) \; z_i \; \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{z_i^2}{2}+z_i\delta\right)} f(c) dz_i dc \tag{A.4}$$

Following [5] we can write the above equation as

$$= e^{-\frac{\delta^2}{2}} \frac{\partial}{\partial \delta} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{z_i^2}{z_i^2+c} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{z_i^2}{2}\right)} \sum_{j=0}^{\infty} \frac{(\delta z_i)^j}{j!} f(c) dz_i dc$$

$$E[\emptyset(z_i^2,c) * z_i] = e^{-\frac{\delta^2}{2}} \frac{\partial}{\partial \delta} \sum_{j=0}^{\infty} \frac{(\delta)^j}{j!} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{z_i^{2+j}}{z_i^2+c} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{z_i^2}{2}\right)} f(c) dz_i dc$$

We notice that the value of integral is zero for odd values of j because then the power of z is odd. Dropping such terms we have

$$E[\emptyset(z_i^2,c) * z_i] = e^{-\frac{\delta^2}{2}} \frac{\partial}{\partial \delta} \left[2 \sum_{j=0}^{\infty} \frac{(\delta)^{2j}}{2j!} \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \int_0^{\infty} \frac{z_i^{2+2j}}{z_i^2+c} e^{\left(-\frac{z_i^2}{2}\right)} f(c) dz_i dc \right] \tag{A.5}$$

Using the duplication formula above expression becomes

$$E[\emptyset(z_i^2,c) * z_i] = e^{-\frac{\delta^2}{2}} \frac{\partial}{\partial \delta} \left[\sum_{j=0}^{\infty} \frac{\left(\frac{\delta^2}{2}\right)^j v^{\frac{v}{2}}}{\Gamma\left(j+\frac{1}{2}\right)\Gamma(j+1)\Gamma\left(\frac{v}{2}\right)} \int_0^{\infty} \int_0^{\infty} \frac{(z_i^2)^{1+j-\frac{1}{2}}}{2^{j+\frac{v}{2}+1/2}(z_i^2+c)} e^{-\left(\frac{z_i^2}{2}+\frac{vc}{2}\right)} (c)^{\frac{v}{2}-1} dz_i^2 \cdot dc \right] \tag{A.6}$$

The integral part

$$I = \int_0^{\infty} \int_0^{\infty} \frac{(z_i^2)^{1+j-\frac{1}{2}}}{2^{j+\frac{v+1}{2}}(z_i^2+c)} e^{-\left(\frac{z_i^2}{2}+\frac{vc}{2}\right)} (c)^{\frac{v}{2}-1} dz_i^2 \cdot dc$$

is computed using the transformations

$$\frac{z_i^2}{z_i^2+c} = U \text{ and } z_i^2 + c = V$$

which gives the integral part as

$$I = v^{-\left(j+\frac{1}{2}+\frac{v}{2}\right)} \sum_{r=0}^{\infty} \left(\frac{v-1}{v}\right)^r \cdot \frac{\Gamma\left(j+\frac{v}{2}+\frac{1}{2}+r\right)}{\Gamma(r+1)} \cdot \frac{\Gamma\left(j+r+\frac{3}{2}\right).\Gamma\left(\frac{v}{2}\right)}{\Gamma\left(j+r+\frac{3}{2}+\frac{v}{2}\right)} \tag{A.7}$$

Substituting the above value of (A.7) and using it in (A.6) we get the first raw moment of $\hat{\beta}_i$ as given in the theorem. Proceeding in the same way the second and higher raw moments of $\hat{\beta}_i$ can be obtained.

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