Research Article

Rings with (x, R, x) in the Left Nucleus

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Abstract If N_l and N_r be the Lie ideals of a nonassociative ring R, then $[N_l, R] \subseteq N_l$ and $[N_r, R] \subseteq N_r$ Also if (x, R, x) is in the left nucleus then $N_l[R, R] \subseteq N_l$. If R is a prime ring with $N_l \neq 0$, and (x, R, x) in the left nucleus then R is either associative or commutative.

Key Word: Nonassociative ring, Left nucleus, Right nucleus, Lie ideals, Associator ideal.

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INTRODUCTION

Kleinfeld [1] studied nonassociative rings with (x, R, x) and $[R, R]$ in the left nucleus. Yen [2] considered the rings with the weaker hypothesis that is, rings with (x, R, x) and $[N_l, R]$ in the left nucleus and proved that if R is a semiprime ring, then $N_r = N_l$. He also proved that if R is a prime ring with $N_l \neq 0$ satisfying one additional condition $N_l[R, R] \subseteq N_l$, then R is either associative or commutative. In this paper by considering N_l and N_r as the Lie ideals of a ring R, we present some properties of R with (x, R, x) in the left nucleus. Using these properties, we show that $N_l[R, R] \subseteq N_l$. Also we prove that, if R is a prime ring with $N_l \neq 0$,then R is either associative or commutative.

PRILIMENARIES

In a nonassociative ring R we define an associator as $(x, y, z) = (xy)\,z - x\,(yz)$ and the commutator as $[x, y] = xy - yx$ for all $x, y, z \in R$. To make the notation more convenient we often use '·' to indicate multiplication as well as juxtaposition. In products, juxtaposition takes precedence, i, e, $xy \cdot z \equiv (xy)\,z$. The nucleus of a ring R is defined as $N = \{n \in R \,/\, (n, R, R) = (R, n, R) = (R, R, n) = 0\}$, the right nucleus as $N_r = \{n \in R \,/\, (R, R, n) = 0\}$ and the left nucleus as $N_l = \{n \in R \,/\, (n, R, R) = 0\}$. A ring R is said to be prime if whenever A and B are ideals of R such that $AB = 0$, then either $A = 0$ or $B = 0$ and is said to be semiprime if for any ideal A of R, $A^2 = 0$ implies $A = 0$. These rings are also refered to as rings free from trivial ideals. And a ring is said to be simple if whenever A is an ideal of R, then either $A= R$ or $A = 0$.

Let R be a nonassociative ring satisfying $(x, R, x) \subseteq N_l$, that is,

$(x, y, z) + (z, y, x) \in N_l$. (1)

Let N_l and N_r be the Lie ideals of R. Then

$[N_l, R] \subseteq N_l$ (2)

and $[N_r, R] \subseteq N_r$.

We use Teichmuller identity which is valid in any arbitrary ring.
$(wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)\, z = 0,$ (3)
for all, $w, x, y, z \in R$.
Then with $w = n \in N_l$ in (3), we obtain
$(nx, y, z) = n(x, y, z)$.
Since N_l is the Lie ideal from (2), we obtain
$(nx, y, z) = n(x, y, z) = (xn, y, z),$ (4)
for all, $n \in N_l$.
Thus N_l is the associative subring of R.

MAIN SECTION

***Lemma* 3.1:** Let $T = \{t \in N_l : t(R, R, R) = 0\}$, then T is an ideal of R.
***Proof*:** In (4) substituting $n = t,$ we obtain
$(tx, y, z) = t(x, y, z) = (xt, y, z) = 0.$
Thus $tR \subset N_l$ and $Rt \subset N_l$.
Also, $tw \cdot (x, y, z) = t \cdot w(x, y, z)$.
Multiplying (3) with t on the left side, we obtain
$t \cdot w(x, y, z) = -\, t \cdot (w, x, y)z$
$= -\, t\, (w, x, y) \cdot z$
$= 0.$
Hence $tw \cdot (x, y, z) = 0$. Thus $TR \subseteq T$.
Now using $TR \subseteq T,$ (2), (4), $RT \subset N_l$ and (1), we obtain
$wt \cdot (x, y, z) = [w, t]\, (x, y, z)$
$= ([w, t]x, y, z)$
$= ((wt)x, y, z) - ((tw)x, y, z)$
$= ([wt, x], y, z) + (x(wt), y, z) - (t(wx), y, z)$
$= ([wt, x], y, z) + (x(wt), y, z)$
$= -\, ((x, w, t), y, z) + ((xw)t, y, z)$
$= -\, ((x, w, t) + (t, w, x), y, z)$
$= 0.$
Hence $RT \subseteq T$. Thus T is an ideal of R. From the definition of $T,$ we obtain $T(R, R, R) = 0$.
This completes the proof of the Lemma.

Let A be the associator ideal of R. We assume that R satisfies (1) and also R is semiprime. Using Lemma 3.1 and equation (3), we obtain $T \cdot A = 0$ and hence $(T \cap A)^2 = 0$. Thus we have $T \cap A = 0$ and $A \cdot T = 0$. (5)
From Lemma 3.1 and equation (3), we obtain
$(R, T, R) = 0.$ (6)

Lemma 3.2: Let R be a nonassociative ring satisfying $(x, y, z) + (z, y, x) \in N_l$. Then $(R, R, N_l) = 0$.
Proof: Let $n \in N_l$, then from (1), we obtain
$(x, y, n) = (x, y, n) + (n, y, x) \in N_l$.
Also from (3), we obtain
$z\, (x, y, n) = (zx, y, n) - (z, xy, n) + (z, x, yn) - (z, x, y)n.$
Hence using these, (4) and (1), we obtain
$(x, y, n)(z, r, s) = (z(x, y, n), r, s)$
$= ((zx, y, n), r, s) - ((z, xy, n), r, s) + ((z, x, yn), r, s) - ((z, x, y)n, r, s)$
$= ((z, x, yn), r, s) - ((z, x, y)n, r, s)$
$= -\, ((yn, x, z), r, s) - (n(z, x, y), \mathrm{r, s})$
$= -\, (n(y, x, z), r, s) - n((z, x, y), r, s)$
$= -\, n((y, x, z), r, s) - n((z, x, y), r, s)$
$= -\, n((y, x, z) + (z, x, y), r, s)$
$= 0.$
Hence $(x, y, n) \in T$.

Since (x, y, n) is also an associator, it is also in A.
Thus from (5), we obtain $(x, y, n) = 0$.
Hence $(R, R, N_l) = 0$.
From Lemma 3.2, we obtain $N_l \subseteq N_r$. (7)

Let $n \in N_r$. Then with $z = n$ in (3), we obtain
$(w, x, yn) = (w, x, y)\, n$. Thus N_r is an associative subring of R.
Now since N_r is the Lie ideal of R, we obtain
$(w, x, yn) = (w, x, y)n = (w, x, ny)$, (8)
for all $n \in N_r$ and $w, x, y \in R$.

***Lemma* 3.3:** Let N_r be the Lie ideal of R and let
$S = \{n \in N_r : (R, R, R)n = 0\}$, then S is an ideal of R, $(R, R, R)S = 0$, $S \cap A = 0$, $S \cdot A = A \cdot S = 0$ and $T \subseteq S$.
***Proof*:** Using (1), (3), (5), (7) and (8) and the proof of Lemma 3.1, this Lemma is proved.

***Lemma* 3.4:** If N_r and N_l are the Lie ideals of R, then $N_r = N_l$ and $S = T$.
***Proof*:** Let us assume that $(R, R, n) = 0$, then from (1), we obtain $(n, x, y) = (n, x, y) + (y, x, n) \in N_l$.
Now using (1), (7), (8) and $[N_r, R] \subseteq N_r$, and since N_r is an associative subring of R, we obtain
$(nx, y, z) - n\,(x, y, z) = \{(nx, y, z) + (z, y, nx)\} - n\{(x, y, z) + (z, y, x)\} + [n, (z, y, x)] \in N_r$.
From the above equation and $(n, R, R) \subseteq N_l \subseteq N_r$ and with $w = n$ in (3), we obtain
$(n, x, y)z = \{(nx, y, z) - n(x, y, z)\} - (n, xy, z) + (n, x, yz) \in N_r$.
Hence using this and (8), we obtain $(s, r, z)\,(n, x, y) = (s, r, (n, x, y)\, z) = 0$, which shows that $(n, x, y) \in S \cap A$ and thus from Lemma 3.3, we have $(n, x, y) = 0$.
Hence $N_r \subseteq N_l$. Thus from (7), we have $N_r = N_l$. From Lemma 3.3 again, $S \cdot A = 0$ and so $S = T$. This completes the proof of the Lemma.

***Theorem* 3.1:** If R is a semiprime ring satisfying $(x, y, z) + (z, y, x) \in N_l$, where N_l is the Lie ideal of R, then T is an ideal of R and $(N_l, R, R) = (R, T, R) = (R, R, N_l) = 0$. Also, if $[N_r, R] \subseteq N_r$, then $N_r = N_l$ and $S = T \subseteq N$.
***Proof*:** From (6) and Lemmas 3.1, 3.2, 3.3 and 3.4 the Theorem is proved.

***Lemma* 2.5:** Let $I = \{a \in R : N_l\, a = 0\}$, then I is an ideal of R.
***Proof*:** First we show that $([R, R], R, R) \subseteq I$. By taking $y = z = x$ in (1), we obtain $(x, x, x) + (x, x, x) = 2(x, x, x) \in N_l$. So $(x, x, x) \in N_l$.
Let $S(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y)$.
Now linearization of (x, x, x) gives $(x, y, z) + (y, z, x) + (z, x, y) + (y, x, z) + (z, y, x) + (x, z, y) \in N_l$.
i.e., $S(x, y, z) + S(y, x, z) \in N_l$. (9)
We have $D(x, y, z) = [xy, z] - x[y, z] - [x, z]y - (x, y, z) - (z, x, y) + (x, z, y) = 0$. (10)
This identity is valid in any arbitrary ring.
Now $D(x, y, z) - D(y, x, z)$ gives
$[[x, y], z] + [[y, z], x] + [[z, x], y] = S(x, y, z) - S(y, x, z)$.
If $z \in N_l$, we obtain $S(x, y, z) - S(y, x, z) \in N_l$. (11)
But from (9), $S(x, y, z) + S(y, x, z) \in N_l$.
i.e., $2S(x, y, z) \in N_l$.
i.e., $S(x, y, z) \in N_l$.
i.e., $(x, y, z) + (y, z, x) + (z, x, y) \in N_l$.
But $(x, y, z), (z, x, y) \in N_l$ implies $(y, z, x) \in N_l$.
i.e., $(R, N_l, R) \subseteq N_l$ implies $((R, N_l, R), R, R) = 0$. (12)
Now in (10) substituting $x = n$ and forming the associators with r, s and using (12), we obtain
$([ny, z], r, s) = (n[y, z], r, s) + ([n, z]\, y, r, s) + ((n, y, z), r, s) + ((z, n, y), r, s) - ((n, z, y), r, s)$
$= (n[y, z], r, s) + ([n, z]y, r, s) + ((z, n, y), r, s)$.
i.e., $([N_lR, R], R, R) = (N_l[R, R], R, R) + ([N_l, R]\, R, R, R) + ((R, N_l, R), R, R)$.
i.e., $(N_l[R, R], R, R) = ([N_lR, R], R, R) - ([N_l, R]\, R, R, R)$

$= ((N_l R)R - R(N_l R) - (N_l R)R + (RN_l)R, R, R)$
$= ((R, N_l, R), R, R)$
$= 0$ from (12).

Thus $N_l([R, R], R, R) = (N_l[R, R], R, R) = 0$. (13)

Hence we have, $([R, R], R, R) \subseteq I$. (14)

Now let $a \in I$, $n \in N_l$ and $x, y, z, w \in R$. Thus we obtain
$n(ax) = (na)x = 0$ implies $IR \subseteq I$.
Now from (13), we obtain
$n(xa) = n[x, a] \in N_l$.
Since $na = 0$ and $n \in N_l$, we obtain $n(a, x, y) = 0$. (15)

Using (15), (1) and since N_l is an associative subring of R, we obtain
$n((yx)a) - n(y(xa)) = n(y, x, a)$
$= n((a, x, y) + (y, x, a)) \in N_l$. (16)

Applying (16) and $n(xa) \in N_l$, we obtain
$n(y(xa)) \in N_l$. (17)

Using (17) and (13), we obtain
$(n(xa))y = n((xa)y)$
$= n[xa, y] + n(y(xa)) \in N_l$.
Combining the above with $n(xa) \in N_l$, we obtain
$n(xa)(y, z, w) = ((n(xa))y, z, w)$
$= 0$.
Hence $n(xa) \in T$ and thus $n(xa) = 0$ implies $RI \subseteq I$.
Therefore I is an ideal of R and thus $NI = 0$.

***Theorem* 3.2:** If N_l is the Lie ideal of a prime ring R with $N_l \neq 0$ and satisfying $(x, y, z) + (z, y, x) \in N_l$, then R is either associative or commutative.

***Proof*:** Since R is prime using (5), we obtain either $A = 0$ or $T = 0$. If $A = 0$, then R is associative. Hence we assume that $T = 0$. Since N_l is the Lie ideal of R, using Lemma 3.2, we see that the ideal of R generated by N_l is $N_l + N_l R$. Then $N_l I = 0$ from Lemma 3.5. Hence we obtain
$(N_l + N_l R)I \subseteq N_l I + (N_l R)I$
$= N_l I + (N_l, R, I) + N_l(RI)$
$\subseteq N_l I + N_l(RI)$
$= 0$.
Thus $[R, R] \subseteq N_l$. Now R satisfies Kleinfeld's hypothesis [1]. Hence it follows that R is either associative or commutative. This completes the proof of the Theorem.

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