Research Article

Two Parameter Inverse Chen Distribution as Survival Model

Pawan Kumar Srivastava[1*], R. S. Srivastava[2**]

[1,2]Department of Mathematics and Statistics, D.D.U. Gorakhpur University, Gorakhpur-273009, INDIA.
Email: [*] chhotu06aug@gmail.com, [**] r_shyam_sri@yahoo.co.in

Abstract A two parameter distribution was revisited by Chen (2000). This distribution can have a bathtub-shaped or increasing failure rate function which enables it to fit real lifetime data sets. In this paper Inverse Chen distribution was introduced**.** Maximum likelihood method used to find Bayes estimator. Also Asymptotic Confidence Intervals, Survival function and Hazard rate of Inverse Chen distribution (ICD) was discussed.
Keywords: Chen Distribution, Inverse Distribution, Maximum Likelihood Estimator (MLE), Survival Function, Hazard Function, Asymptotic Confidence Intervals etc.

[*]**Address for Correspondence:**
Dr. Pawan Kumar Srivastava, Department of Mathematics and Statistics, D.D.U. Gorakhpur University, Gorakhpur-273009, INDIA.
Email: chhotu06aug@gmail.com
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INTRODUCTION

Chen (2000) proposed a new two parameter lifetime distribution with bathtub shaped or increasing failure rate (IFR) function. Some probability distributions have been proposed with modals for bathtub-shaped failure rates, such as Hjorth (1980), X i.e. and Lai (1996). The new two parameter lifetime distribution with bathtub-shaped or increasing failure rate function compared with other models has some useful properties. It is observed that the lifetime distribution of many electronic, mechanical and electro mechanical products often has non monotone failure rate functions. In many reliability analysis especially over the life-cycle of the product, it usually involves high initial failure rates (infants mortality) and eventual high failure rates due to aging and indicating a bathtub shape failure rate. For many electro mechanical, electronic and mechanical products, the failure rate function has a bathtub-shaped curve. It includes three phases: early failure phase with a decreasing failure rate, normal use phase with an approximately constant failure rate, and wear-out phase with an increasing failure rate models which allow only monotone failure rates might not be appropriate or adequate for modeling the whole bathtub-shaped data. Hence, some probability distributions have been proposed to fit real life data with bathtub-shaped failure rates, such as Gaver and Acar (1979), Rajarshi and Rajarshi (1993), Wang (2000), Xai *et al*.(2002) and Wu *et al*. (2004). Many parametric probability distributions have been introduced to analyze sets real data with bathtub-shapes failure rates. The bathtub-shape hazard function provides an appropriate conceptual modal for some electronic and mechanical products as well as the lifetime of humans. The previous work in detail on parametric probability distributions with bathtub-shaped failure rate function can be referred to many different authors papers. Researchers got interested in distributions with non-monotone hazard function, such as bathtub-shape and unimodal hazard functions and noticed that distributions with one or two parameter like the Weibull distributions have very strong restrictions on the data. Smith and Bain (1975) gave the exponential power distribution whose hazard function has a bathtub shape. Mudholkar and Srivastava (1993) provided an Exponentiated-Weibull distribution. This distribution has monotone increasing, monotone decreasing, bathtub or unimodal failure rate depending on the different parameter ranges. Chen (2000) proposed a two parameter lifetime distribution with bathtub-shape or increasing hazard function. Its cumulative distribution function (CDF) is given by

$F_C(x) = 1\text{-}e^{\lambda(1-e^{x^\beta})}$, (x>0, λ, β>0) (1.1)

And hence the probability distribution function (PDF) is given by

f(x) = λ β $(x)^{\beta-1}\ e^{x^\beta}\ e^{\lambda(1-e^{x^\beta})}$, (x>0, λ, β>0) (1.2)

If a random variables X has a Chen distribution, then the distribution of Y=$\frac{1}{X}$ may be termed as an inverse Chen distribution (ICD). Its cumulative distribution function (CDF) is define by

$F(Y) = P(Y \leq y) = P\left(\frac{1}{X} \leq y\right)$

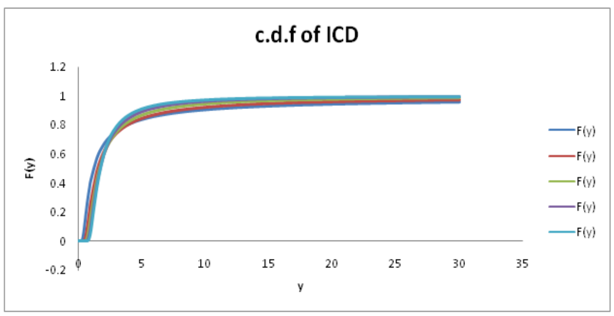
$= P\left(\frac{1}{y} \leq Y\right) = P\left(X \geq \frac{1}{y}\right)$

$=1- P\left(X < \frac{1}{y}\right)$

$= 1- F_C\left(\frac{1}{y}\right)$

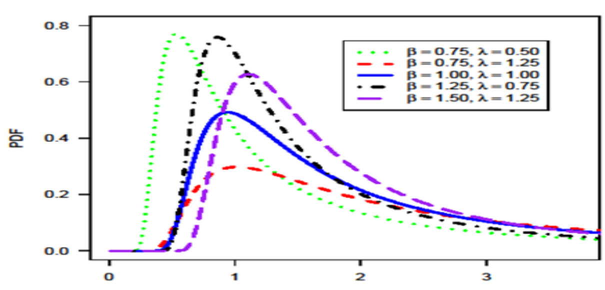
$= 1- \left(1\text{-}e^{\lambda(1-e^{y^{-\beta}})}\right)$

$F(y) = e^{\lambda(1-e^{y^{-\beta}})}$, (y>0, λ, β>0) (1.3)



And the probability density function (pdf) of Inverse Chen distribution (ICD) is

f (y) = $\lambda\, \beta\, y^{-(\beta+1)}\ e^{[y^{-\beta}+\lambda(1-e^{y^{-\beta}})]}$, (y>0, λ, β>0)
(1.4)



SURVIVAL FUNCTION

The object of primary interest is the survival function, conventionally denoted by *S*, which is defined as

S(t) = Pr(T>t) (2.1)

where *t* is time, *T* is a random variable denoting the time of death, and "Pr" stands for probability. That is, the survival function is the probability that the time of death is later than some specified time *t*. The survival function is also called the survivor function or survivorship function in problems of biological survival, and the reliability function in mechanical survival problems. In the latter case, the reliability function is denoted *R*

(t). Usually one assumes S(0) = 1, although it could be less than 1 if there is the possibility of immediate death or failure. The survival function can be expressed in terms of probability distribution and probability density functions

S(t) = Pr(T>t) = $\int_t^\infty f(u)$du = 1- F(t) (2.2)

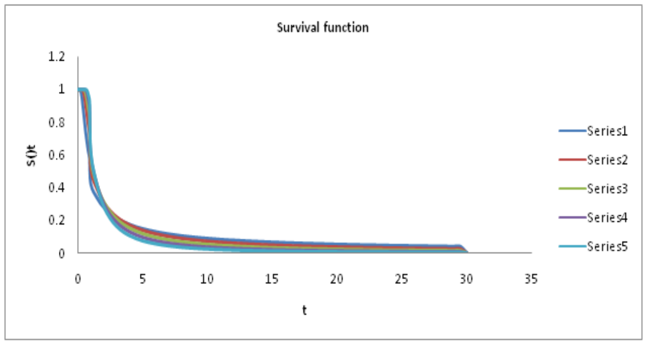
Similarly, a survival event density function can be defined as

S'(t) = $\frac{d}{dt}$S(t) = $\frac{d}{dt}\int_t^\infty f(u)$du =$\frac{d}{dt}$ [1- F(t)] = −f(t) (2.3)

Now the survival function of Inverse Chen Distribution (ICD) is

S(t) = 1- F(t)

S(t) = 1- $e^{\lambda(1-e^{t^{-\beta}})}$ (2.4)



HAZARD FUNCTION

The hazard function, conventionally denoted λ, is defined as the event rate at time t conditional on survival until time t or later (that is, $T \geq t$),

λ (t) =$\lim_{dt\to 0}\frac{\Pr(t\leq T<t+dt|T\geq t)}{dt.S(t)} = \frac{f(t)}{S(t)}$ = - $\frac{S'(t)}{S(t)}$ (3.1)

Force of mortality is a synonym of hazard function which is used particularly in demography and actuarial science, where it is denoted byμ. The term hazard rate is another synonym. The hazard function must be non-negative, λ (t) ≥ 0, and its integral over [0, ∞] must be infinite, but is not otherwise constrained; it may be increasing or decreasing, non-monotonic, or discontinuous. An example is the bathtub curve hazard function, which is large for small values of t, decreasing to some minimum, and thereafter increasing again; this can model the property of some mechanical systems to either fail soon after operation, or much later, as the system ages. The hazard function can alternatively be represented in terms of the cumulative hazard function, conventionally denoted:

(t) = - log S(t) (3.2)

so transposing signs and exponentiation

S(t) = exp (-(t)) (3.3)

or differentiating (with the chain rule)

$\frac{d}{dt}$(t) = - $\frac{S'(t)}{S(t)}$ = λ(t) (3.4)

The name "cumulative hazard function" is derived from the fact that

(t) = $\int_0^t \lambda$(u) du (3.5)

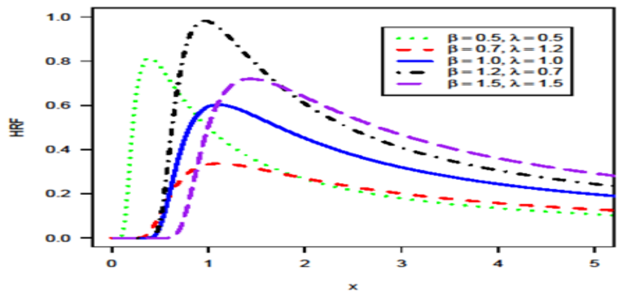
which is the "accumulation" of the hazard over time.

From the definition of (t), we see that it increases without bound as t tends to infinity (assuming that $S(t)$ tends to zero). This implies that λ(t) must not decrease too quickly, since, by definition, the cumulative hazard has to diverge. For example, exp(-t) is not the hazard function of any survival distribution, because its integral converges to1.

Now the Hazard function of Inverse Chen Distribution (ICD) is given by

h (t) = $\frac{f(t)}{S(t)}$

$$h(t) = \frac{\lambda\,\beta\, t^{-(\beta+1)}\ e^{[t^{-\beta}+\lambda(1-e^{t^{-\beta}})]}}{1-e^{\lambda(1-e^{t^{-\beta}})}} \tag{3.6}$$



MAXIMUM LIKELIHOOD ESTIMATION

The probability density function (pdf) of Inverse Chen distribution (ICD) is

$$f(y) = \lambda\,\beta\, y^{-(\beta+1)}\ e^{[y^{-\beta}+\lambda(1-e^{y^{-\beta}})]}, \ (y>0,\ \lambda,\ \beta>0) \tag{4.1}$$

In this section we use the maximum likelihood method to estimate the two unknown parameters λ and β. Suppose $X_1, X_2, \dots, X_n$ is a random sample from ICD (λ, β), then the likelihood function of the observed data is

$$L = (\lambda\beta)^n\ [\prod_1^n y_i]^{-(1+\beta)}\ e^{\sum_1^n\{y_i^{-\beta}+\lambda(1-e^{y_i^{-\beta}})\}} \tag{4.2}$$

The log likelihood becomes

$$L^* = n\,(\log\lambda + \log\beta) - (1+\beta)\sum_{i=1}^n \log\ y_i + \sum_{i=1}^n [y_i^{-\beta} + \lambda(1-e^{y_i^{-\beta}})] \tag{4.3}$$

The corresponding likelihood equations are

$$\frac{\delta L^*}{\delta\lambda} = \frac{n}{\lambda} + n - \sum_{i=1}^n e^{y_i^{-\beta}} = 0 \tag{4.4}$$

And

$$\frac{\delta L^*}{\delta\beta} = \frac{n}{\beta} - \sum_{i=1}^n log y_i\ - \sum_{i=1}^n y_i^{-\beta} + \lambda\sum_{i=1}^n e^{y_i^{-\beta}}\ y_i^{-\beta}\ log y_i\ = 0 \tag{4.5}$$

From (4.4) we get the MLE of λ

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^n e^{y_i^{-\beta}} - n} \tag{4.6}$$

From (4.6), $\hat{\beta}$ is the solution of the following non linear equation

$$\frac{n}{\beta} - \sum_{i=1}^n log y_i\ - \sum_{i=1}^n y_i^{-\beta} + \frac{n\sum_{i=1}^n e^{y_i^{-\beta}}\ y_i^{-\beta}\ log y_i}{\sum_{i=1}^n e^{y_i^{-\beta}} - n} = 0 \tag{4.7}$$

A closed form solution of (4.7) does not exist, so a numerical technique must be used to find the maximum likelihood estimate of β for any given set.

APPROXIMATE CONFIDENCE INTERVAL

The exact distribution of MLEs cannot be obtained explicitly. Therefore, the asymptotic properties (Chaudhary A.K. and Kumar V. (2014).) of MLEs can be used to construct the confidence intervals for the parameters. Under some regularity conditions, the MLEs

$$\hat{\theta} = (\hat{\lambda}, \hat{\beta}) \rightarrow N_2\ (0, (I(\theta))^{-1}) \tag{5.1}$$

Where I(θ) is the variance matrix. As I (θ) involves the unknown parameter, we replace these parameter by their corresponding MLEs to obtain an estimate I($\hat{\theta}$)

$$I\,(\hat{\theta}) = - \begin{bmatrix} \frac{\delta^2 L^*}{\delta\lambda^2} & \frac{\delta^2 L^*}{\delta\lambda\delta\beta} \\ \frac{\delta^2 L^*}{\delta\beta\delta\lambda} & \frac{\delta^2 L^*}{\delta\beta^2} \end{bmatrix}_{(\lambda=\hat{\lambda}, \beta=\hat{\beta})} \tag{5.2}$$

Where

$$\frac{\delta^2 L^*}{\delta\lambda^2} = -\frac{n}{\lambda^2} \tag{5.3}$$

and

$\frac{\delta^2 L^*}{\delta\beta^2} = -\frac{n}{\beta^2} + \sum_{i=1}^{n}(log y_i)^2 ({y_i}^{-\beta})$ - λ$\sum_{i=1}^{n}[(log y_i)^2 e^{{y_i}^{-\beta}}\ {y_i}^{-\beta}({y_i}^{-\beta}+1)]$ (5.4)

$\frac{\delta^2 L^*}{\delta\lambda\delta\beta} = \sum_{i=1}^{n} e^{{y_i}^{-\beta}}\ {y_i}^{-\beta}\ log y_i$ (5.5)

And

$\frac{\delta^2 L^*}{\delta\beta\delta\lambda} = \sum_{i=1}^{n} e^{{y_i}^{-\beta}}\ {y_i}^{-\beta}\ log y_i$ (5.6)

The diagonal elements of $(\mathrm{I}(\theta))^{-1}$ provide the asymptotic variance for the parameters λ, β respectively. The 100 (1- α) % confidence intervals for λ, β can be constructed as

$\hat{\lambda} \pm z_{\propto/2}$) $\sqrt{\mathrm{var}(\hat{\lambda})}$

and $\hat{\beta} \pm z_{\propto/2} \sqrt{\mathrm{var}(\hat{\beta})}$ (5.7)

where $z_{\propto/2}$ is the upper percentile of standard normal variate.

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