Research Article

Generalization of Fixed Point Theorems

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Abstract The fixed point theory is an important and core topic of the non-linear analysis and concerned with the study of the functional equation in metric spaces. In this paper, we give generalization of some fixed theorems. The main theme of this paper is the generalization of the Nadler's fixed point theorem.
Key words: fixed point theory, metric spaces.

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INTRODUCTION

In this Paper we prove generalization of Nadler's fixed point theorem. Let (X, d) be a metric space CB(X) denotes the collection of all nonempty closed bounded subset of X.

For A.B ∈ CB(X) and x ∈ X

Define

$D(x, A) = \inf \{d(x, a)\ a \in A\}$ and

$H(A, B) = \max \{\sup_{a \varepsilon A} D(a, B), \sup_{b \varepsilon B} D(b, A)\}$

It is easy to see that H is a metric on CB(X). H is called the Hausdorff metric induced by d.

Many fixed point theorems have been proved by various authors as generalization to Banach's Contraction Mapping Principle one such generalization is due to Geraghty [1] is given in this paper.

DEFINITION 1.1

An element x ∈ X. is said to be a fixed point of a multi-valued mappings. $T : X \longrightarrow CB(X)$ if such that $x \in T(X)$. One can show that [CB(X), H] is a complete metric space.

In 1969 Nadler [2] extended the Banach Contraction Principle to set valued mapping.

In this among other things, we give generalizations of Nadler's fixed point theorem. The following Lemma has important role in the proof of main theorem.

LEMMA 1

Let (X, d) be a metric space and A, B ∈ CB(X). Then for each a ∈ A and ∈ > 0, there exist an b ∈ B such that $d(a, b) \leq H(A, B) + \in$.

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MAIN RESULT

We start our work with our main result which can be regarded as an extension of extension of Nadler's Fixed Point Theorem.

THEOREM 1

Let (X, d) be a complete metric space and T be a mapping from X into CB(X) such that
$H(Tx, Ty) \leq \alpha d(x, y) + \beta[D(x, Tx) + D(y, Ty)] + \gamma[D(x, Ty) + D(y, Tx)]$
For all $x, y \in X$. Where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.

PROOF

Let $x_0 \in X$, $x_1 \in Tx_0$ and define

$$r = \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)}$$

If r = 0. The proof is clear. Now Assume r > 0
Then it follows from lemma (1) that

$\ni x_2 \in Tx_1 \quad d(x_1, x_2) \leq H(Tx_0, Tx_1) + r$
$\ni x_3 \in Tx_2 \quad d(x_2, x_3) \leq H(Tx_1, Tx_2) + r^2$
:
:
$\ni x_{(n+1)} \in Txn_2 \quad d(x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_n) + r^n$

Hence we have,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq H(Tx_{n-1}, Tx_n) + r^n \\ &\leq \alpha\, d(x_{n-1}, x_n) + \beta[D(x_n, T x_n) + D(x_{n-1}, Tx_{n-1})] \\ &\quad + \gamma[D(x_n, Tx_{n-1}) + D(x_{n-1}, Tx_n)] + r^n \\ &\leq \alpha\, d(x_{n-1}, x_n) + \beta[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\ &\quad + \gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + r^n \end{aligned}$$

For all $n \in N$.

it follows that, $d(x_n, x_{n+1}) \leq r\, d(x_{n-1}, x_n) + \frac{r^n}{1-(\beta+\gamma)}$

For all $n \in N$,

it can be conclude that

$$d(x_n, x_{n+1}) \leq r^n\, d(x_0, x_1) + \frac{n r^n}{1-(\beta+\gamma)}$$

For all $n \in N$, Now since $r < 1$, then

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$$

It follows that $\{ x_n \}$ is a Cauchy sequence in X.
By completeness of X there exist $x^* \in X$ such that,

$$\lim_{n\to\infty} x_n = x^*$$

We are going to show that x^* is a fixed point of T.
We have,

$$\begin{aligned} D(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + D(x_{n+1}, Tx^*) \\ &\leq d(x^*, x_{n+1}) + H(x_{n+1}, Tx^*) \\ &\leq d(x^*, x_{n+1}) + \alpha\, d(x_n, x^*) + \beta\, [D(x_n, Tx_n) + D(x^*, Tx^*)] \\ &\quad + \gamma[D(x_n, Tx^*) + D(x_n, Tx_n)] \end{aligned}$$

For all $n \in N$, therefore

$$\begin{aligned} D(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + \alpha\, d(x_n, x^*) + \beta\, [d(x_n, x_{n+1}) + D(x^*, Tx^*)] \\ &\quad + \gamma[D(x_n, Tx^*) + d(x_{n+1}, x^*)] \qquad \ldots\ldots (1.1) \end{aligned}$$

For all $n \in N$, Passing the $\lim_{n\to\infty}$ in (1.1) then we have,

$$D(x^*, Tx^*) \leq (\beta + \gamma)\, D(x^*, Tx^*)$$

On other hand, $\beta + \gamma < 1$, then

$$D(x^*, Tx^*) = 0$$

It follows that $x^* \in Tx^*$.

CORLLARY 1

Let (X, d) be a complete metric space and Let T be a mapping from X. into X, such that

$D(Tx, Ty) \leq \alpha\, d(x, y) + \beta[d(x, Tx)+d(y, Ty)] + \gamma[d(x, Ty)+d(y, Tx)]$

For all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and

$\alpha + 2\beta + 2\gamma < 1$.

Then T has a fixed point.

COROLLARY 2

Let (X, d) be a complete metric space and Let T be a mapping from (X, d) into [CB(X), H], satisfies

$H(Tx, Ty) \leq a_1\, d(x, y) + a_2\, D(x, Tx) + a_3\, D(y, Ty) + a_4 D(x, Ty) + a_5\, D(y, Tx)$

For all $x, y \in X$, where $a_i \geq 0$ and for each $i \in \{1, 2, \ldots 5\}$

$\sum_{1=1}^{5} a_i < 1$ Then T has a fixed point.

COROLLARY 3

Let (X, d) be a complete metric space and let T be a mapping from (X, d) into [CB(X), H] satisfies

$H(Tx, Ty) \leq \alpha\, d(x, y)$

For all $x, y \in X$ where $0 \leq \alpha < 1$.

Then T has a fixed point.

COROLLARY 4

Let (X, d) be a complete metric space and let T be a mapping from (X, d) into [CB(X), H] satisfies

$H(Tx, Ty) \leq \beta\, [D(x, Tx) + D(y, Ty)]$

For all $x, y \in X$ where $\beta \in \left(0, \frac{1}{2}\right)$.

Then T has a fixed point.

COROLLARY 5

Let (X, d) be a complete metric space and let T be a mapping from (X, d) into [CB(X), H] satisfies

$H(Tx, Ty) \leq \gamma\, [D(x, Ty) + D(y, Tx)]$

For all $x, y \in X$ where $\gamma \in \left(0, \frac{1}{2}\right)$. Then T has a fixed point.

COROLLARY 6

Let (X, d) be a complete metric space and let T be a mapping from (X, d) into [CB(X), H] satisfies

$H(Tx, Ty) \leq \alpha\, d(x, y) + \beta\, [D(x, Tx) + D(y, Ty)]$

For all $x, y \in X$ where $\alpha + 2\beta < 1$

Then T has a fixed point.

THEOREM 2

Let (X, d) be a complete metric space and let $f : x \longrightarrow X$ be a mapping such that for each $x, y \in X$

$D[f(x), f(y)] \leq \alpha[d(x, y), d(x.y)]$

Where α is a function from $[0, \infty)$ into $[0, 1)$ which satisfy the simple condition $\alpha(tn) \longrightarrow 1$.

$\Rightarrow tn \longrightarrow 0$ then F has a fixed point.

$Z \in X$ and $\{f^{n}(x)\}$ converges to Z.

For each $x \in X$

PROOF

Let (X, d) be a metric space. Let CB(X) denotes the collection of all non empty closed bounded subset of X. for $A, B \in CB(X)$ and $x \in X$

$D(x, A) = \inf \{d(x,a)\ a \in A\}$

$Hd(A, B) = \max \{\sup_{a \varepsilon A} D(a, B), \sup_{b \varepsilon B} D(b, A)\}$

It is easy to see that Hd is a metric on CB(X).

Hd is called the Hausdorff metric induced by d. A point $P \in X$ is said to be a fixed point of multi-valued mapping.

$T : x \longrightarrow CB(X)$ if $P \in T(p)$

The fixed point theory of multi-valued contraction was initiated by Nadler [2] in the following way.

THEOREM 3

Let (X, d) be a complete metric space and let T be a mapping from X into CB(X) such that for all $x, y \in X$

$Hd(Tx,Ty) \leq rd(x, y)$

Where $0 \leq r < 1$, Then T has a fixed point.

THEOREM 4

Let (X, d) be a complete metric space and let T be a mapping from X into CB(X). Assume

$$Hd(Tx, Ty) \leq \alpha\,[d(x, y), d(x,y)]$$

For all x, y ∈ X, where α is a function from [0, ∞) into [0, 1) satisfying

$$\limsup_{s \to t^t} \alpha(s) < 1$$

For all t ∈ [0, ∞), then T has a fixed point.

Recently Eldved et.al [4] claimed that Nadler's fixed point theorem is equivalent to Mizoguch and Takahashi's Fixed Point Theorem. Very recently Suzuki's [7] produced an example to disproved their claim and showed that Mizoguchi and Takahashi's Fixed Point Theorem is a real generalization of Nadler's theorem.

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