Research Article

The boundedness of solution of nonlinear third order differential equations

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Abstract This paper presents one such technique for the solution of a class of ordinary nonlinear differential equations. The technique is capable of deriving closed-form descriptions of the qualitative temporal behaviour represented by such equations. The boundedness of solutions of certain nonlinear third- order delay differential equations. Sufficient conditions for the boundedness of solutions for the equations considered are obtained by constructing a Lyapunov functional.
Keywords: nonlinear, differential equations.

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INTRODUCTION

More recently, qualitative phase-space approaches have been introduced [Lee and Kuipers 88, Sacks 87]. Augmenting simulation, these techniques explore trajectories in phase space, showing how the qualitative values in a system will change from any point in the space. Similar to the phase-space methods used in quantitative analysis (Thompson and Stewart 86), these techniques are strong at indicating convergence, stability, etc. But weaker at explicitly describing the temporal behaviour of the values. Such qualitative solutions to differential equations are desirable for several reasons. First, if an exact solution can indicate the types of behaviour that are possible, augmenting numerical simulation results. Also, for complex equations where an exact solution is known, it may be so complex as to not be comprehensible solution may be preferable for obtaining an intuitive understanding of system behaviour. The advantages of qualitative descriptions of behaviour are covered further in (Yip 88). We should recognize that some significant theoretical results concerning the stability and boundedness of solutions third order nonlinear differential equations with delay have been achieved, see for example the papers of Sadek[5], Tejumola and Tchegnani[6], Tunc[7,8] Zhu[9] and the references citied in these papers. It should be noted that, in 1969, Palusinski *et al.*[10] applied an energy metric algorithm for the generation of a Lyapunov function for third order ordinary nonlinear differential equation of the form:

$x''' + a_1 x'' + f_2(x')x' + a_3 x = 0$

They found some conditions for the stability of zero solution of this equation as follows:

$a_1 > 0, f_2(x') > a_3 > 0.$

In this paper we are concerned with the third ordinary nonlinear delay differential equations of the type

$$x'''(t) + a_1 x''(t) + f_2(x'(t-r(t)) + a_3 x(t) = p(t, x(t), x(t), x(tr(t)), x'(t-r(t)), x''(t)) \rightarrow (1)$$

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or its equivalent system

$x'(t)=y(t),\ y'(t)=z(t),$

$$z'(t) = -a_1 z(t) - f_2(y(t)) - a_3 x(t) + \int_{t-r(t)}^{t} f_2'(y(s))z(s)ds$$

$+p(t,x(t), y(t), x(t-r(t)), y(t-r(t)),z(t)),$

where r is a bonded delay, $0\leq r(t) \leq \gamma, r'(t) \leq \beta, 0 < \beta < 1,$

β *and* γ are some positive constants γ which will be determined later;

a_1 *and* a_3 are some positive constants; the functions f_2 and p depend only on the arguments displayed explicitly and the primes in equation (1) denote differentiations with respect to t

THEOREM

In addition to the basic assumptions imposed on the functions f_2 and p that appeared in equation.

$x'''(t) + a_1 x''(t) + f_2(x'(t - r(t)) + a_3 x(t) =$

$P\left(t, x(t), x'(t), x(t - r(t)), x'(t - r(t)), x''(t)\right) \rightarrow$ (A)

We assume that there are positive Constants , $a_1, a_2, a_3, \varepsilon_0, L, \mu$ HandH_1, such that the Condition satisfied for x, y, z in Ω: = {(x, y, z) Є R^3:|x|< H_1, |y|<H_1,|z|<H_1, H_1<H }

1. $a_1, a_2, -a_3 > 0,\ f_2(0) = 0, \frac{f_2(y)}{y} - a_2 >, \ \varepsilon_0, (y \neq 0)$ and $|f_2^1(y)| \leq L$
2. $|\, p(t, x(t), y(t), x(t-r(t)), y(t-r(t)), z(t)| \leq q(t),$

Where $\max q(t) < \infty$ and $q \in L^1(0,\infty)$

The space of integrable lebesgue function than there exist a finite positive Constant k_1 such that the solution x(t)of equation (A)difined by the initial functions.

$x(t) = \emptyset(t), x^1(t) = \emptyset'(t), x''(t) = \emptyset''(t)$

Satisties the in equalities

$|x(t)| \leq k_1, |x'(t)| \leq k_1, |x''(t)| \leq k_1$

For all $t \geq t_0$, where $\emptyset \in c^2([t_0-r,t_0],R)$

Provided that

$\gamma < \min\left\{\frac{2\varepsilon_0}{L}, \frac{2(a_1 a_2 - a_3)}{a_2 L + 2\mu}\right\}$

Proof: see [13,PP84]. If f(t, $\emptyset$) in x= f(t, x(t)),
xt (ɵ)= x(t+ɵ), - r ≤ɵ ≤ 0 ,t ≥0 is continuous in t, ø, for every ØЄC H,H_1<H andt_0 ,0≤ t_0 <c, where C is (+)ve Constant then there exist a solution with initial value Ø at t= t_0 and this solution continues for t>t_0, Now. The proof of this theorem also depends on the Scalar. Differentiable lyapunov functional V=V (x(t), y(t), z(t)) defined in

$V(x(t),y(t),z(t)) = \frac{1}{2}a_3^2x^2 + a_2a_3xy + \frac{1}{2}a_2z^2 + a_3yz +$

$a_2\int_0^y f_2(\xi)\, d\xi + \frac{1}{2}\, a_1\, a_3 y^2 + \mu \int_{-r(t)}^{0}, \int_{t+s}^{t} z^2(\theta)\, d\theta\, d\, s$

Now since, $P(t, x(t), y(t), x(t - r(t)), y(t - r(t), z(t)) \neq\ 0$ in view of

$Z'(t) = -a_1 z(t) - f_2(y(t)) - a_3x(t) + \int_{t-r(t)}^{t} f_2^1(y(s)z(s))^2 ds + p(t, x(t), y(t), x(t - r(t)), y(t - r(t)), z(t))$ and

$\frac{d}{dt}v(x(t), y(t), z(t)) \leq -\propto y^2 - \rho z^2$

It can be satisfies following inequality.

$\frac{d}{dt}V(x(t), y(t), z(t)) \leq -\propto y^2 - \rho z^2 + |a_3 y + a_2 z|.\left|P\left(t, x(t), y(t), x(t - r(t)), y(t - r(t)), y(t)\right)\right|$

$\leq -\propto y^2 - \rho z^2 - |a_3 y + a_2 z\, |q(t)$ Hence it follows that

$\frac{d}{dt}V(x(t), y(t), z(t) \leq -\propto y^2 - \rho z^2 + D_2(|y| + |z|)q(t)$

Hence if follows that

$\frac{d}{dt}V(x(t), y(t), z(t)) \leq D_2(|y| + |z|)q(t)$

for a Constant $D_2 > 0$ *where*, $D_2 = \max\{1, a^{-1}\}$

making use of inequalities

$|y| < 1 + y^2$ and $|z| < 1 + z^2$

it is clear that

$\frac{d}{dt}V(x(t), y(t), z(t)) \leq (|y| + a^{-1}|z|)q(t)$

$\leq D_2(|y| + |z|)q(t)$

$\leq D_2(2 + y^2 + z^2)q(t)$

we know

$V(x(t), y(t), z(t)) \geq D^1\ (x^2 + y^2 + z^2)\ \rightarrow (1)$

we *have*,

$(x^2 + y^2 + z^2) \leq D_1^{-1}V(x(t), y(t), z(t))$

hence,

$\frac{d}{dt}V(x(t), y(t), z(t)) \leq D_2(2 + D_1^{-1}V(x(t), y(t), z(t))q(t)$

$\frac{d}{dt}V(x(t), y(t), z(t)) \leq 2D_2q(t) + D_2D_1^{-1}V(x(t), y(t), z(t))q(t)$

Now, integrating the last inequality from 0 *to t using the assumption* $q\epsilon L(0, \infty)$ and *Gronwall − Reid − Bellman inequality we obtain*

$V(x(t), y(t), z(t)) \leq V(0,0,0) + 2D_2E + D_2D_1^{-1} \int_0^t (V(x(s), y(s), z(s))q(s)ds$

$\leq V(0,0,0) + 2D_2E) exp\ (D_2D_1^{-1} \int_0^t q(s)ds)$

$\leq V(0,0,0) + 2D_2E)\ exp(D_2D_1^{-1}\ E) = k_2 < \infty\ \rightarrow (2)$

where $k_2 > 0$ *is a constant*

$k_2 = (V(0,0,0) + 2D_2E)\ exp(D_2D_1^{-1}\ E)$

and E=$\int_0^\infty q(s)ds$

Now the inequalities (1)and(2)

We get

$x^2(t) - y^2(t) + z^2(t) \leq D_1^{-1}V(x(t), y(t), z(t) \leq k_3$

Where,$k_3 = k_2\ D_1^{-1}$

we Conclude that

$|x(t)| \leq k_3, |y(t)| \leq k_3, |z(t)| \leq k_3$

for all $t \geq t_0$

Hence,

$|x(t)| \leq k_3, |x^{'}(t)| \leq k_3, |x^{''}(t) \leq k_3$

for all $t \geq t_0$

thus the proof of theorem is now Complete.

Example.

consider the third order nonlinear delay differential equation.

$$x'(t)+[x^2(t)+x'(t)+4]x''(t) + 8x^{'}(t - r(t))$$
$$+ \sin x^{'}(t - r(t)) + \frac{x(t - r(t)}{1 + x^2(t - r(t))} \rightarrow (1)$$
$$= \frac{4}{1+t^2+x^2(t)+x'^2(t)+x^2(t-r(t))+x'^2(t-r(t))+x''^2(t)}$$

or its equivalent system form

$$x' = y,$$
$$y^{'} = z,$$
$$z^{'} = -[y^2 + y + 4]z - [8y + \sin y] - \frac{x}{1 + x^2}$$
$$+ \int_{t-r}^{t}(8 + \cos y\ (s))z(s) + \int_{t-r(t)}^{t} \frac{1 - x(s)}{(1 + x^2(s))^2} y(s)ds$$
$$+ \frac{4}{1 + t^2 + x^2 + y^2 + x^2(t - r(t)) + y^2(t - r(t)) + y^2(t)}$$

observe that

$$\frac{4}{1+t^2+x^2+y^2+x^2(t-r(t))+y^2(t-r(t))+z^2} \leq \frac{4}{1+t^2} = q(t)$$

for all $t \in R^+, x, y, x(t - r(t)), y(t - r(t)), z$ *and*

$\int_0^\infty q(s)ds = \int_0^\infty \frac{4}{1+s^2}\,\mathrm{ds} = \pi < \infty, that\ is\ q\epsilon L^1(0,\infty)$
$to\ show\ the\ boundedness\ of\ solutions\ we\ use\ as\ a\ main\ tool\ the$
$Lyapunov\ functional\ now, in\ view\ the\ \mathrm{time}\ derivative\ of\ the$
$functional\ Vx(t), y(t), z(t))with\ respect\ to\ the\ system\ can\ be\ revised\ as$
$follows\text{: } \frac{d}{dt} Vx(t), y(t), z(t)) \le$

$$-vy^2 - pz^2 + \frac{y + a^{-1}z}{1 + t^2 + x^2 + y^2 + x^2(t - r(t)) + y^2\big(t - r(t)\big) + z^2}$$

$Making\ use\ of\ the\ fact$

$$\frac{1}{1+t^2+x^2+y^2+x^2(t-r(t))+y^2\big(t-r(t)\big)+z^2} \le \frac{1}{1+t^2}$$

$we\ get$

$$\frac{d}{dt}\ V\ x(t), y(t), z(t))\ \le -Vy^2 - \rho z^2 + \frac{4|y+a^{-1}z|}{1+t^2}$$

$Hence\ it\ is\ obvious\ that$

$$\frac{d}{dt}\ V\ x(t), y(t), z(t))\ \le \frac{4|y+z|}{1+t^2} \le \frac{4|y|+|z|}{1+t^2}$$

$$\le \frac{4(4+y^2+z^2)}{1+t^2} = \frac{8}{1+t^2} + \frac{4(y^2+z^2)}{1+t^2} \to (2)$$

$\le \frac{8}{1+t^2} + \frac{4D_1^{-1}}{1+t^2} V\ x(t), y(t), z(t))\ Now, integrating\ (2) from\ 0\ to\ t_1\ using\ the\ fact\ \frac{1}{1+t^2}\ \epsilon L^1(0,\infty) and$
$andGronwall - Reid - Bellman\ inequality, it\ \mathrm{can}\ be\ easily$
$concluded\ the\ boundedness\ of\ all\ solutions\ of\ (1).$

REFERENCES

1. Lee, W.L. and Kuipers. B.J., 1988, “Non- Intersection of Trajectories in Qualitative phase Space: A Global Constranint for Qualitative simulation,” AAAI-88, pp. 286-290.
2. Sacks, Elisha, 1987, “piecewise Linear Reasoning.” AAAI-87, pp, 655-659.
3. Thompson, J.M.T., H.B. Stewart, 1986, Nonlinear Dynamics and Chaos, John Wiley.
4. Yip, K.M., 1988, “Generating Global Behaviors Using Deep Knowledge of Local Dynamics,” AAAI-88, pp. 280-285.
5. A.I. Sadek, Stability and boundedness of a kind of third-order delay differential system, Applied Mathematics Letters. 16(5), (2003),657-662
6. H.O. Tejumola and B. Tchegnani, Stability, boundedness and existence of periodic solutions of some third and fourth order nonlinear delay differential equations. J. Nigerian Math. Soc.19,(2000),9-19.
7. C. Tunc, New results about stability and boundedness of solutions of certain non-linear third-order delay different equations. The Arabian Journal for science and Engineering, Volume 31, Number 2A,(2006),185-196.
8. C. Tunc, On the bundedness of solutions of delay differential equations of third order, Differetial’ niye Uravneniya (Differential Equations). (2007), (accepted for publication).
9. Y.F. Zhu, On stability, boundedness and existence of periodic solution of a kind third order nonlinear deley differential system. Ann. Differrential Equations 8 (2), (1992), 249-259.
10. O. Palusinski,; P. Stern Wall, ; M. Moe, Comments on “An energy metric algorithm for the generation of Liapunov functions” . IEEE Transactions on Automatic Control, Volume 14, Issue 1, (1969), 110-111.
11. T. Yoshizawa, Stability theory by Liapunov’s second method. The Mathematical Society of Japan, Tokyo, 1966.
12. A.U. Afuwape and M.O. Omeike, On the stability and boundedness of solutions of a.kind of third order delay differential equations. Appl. Math. Comput., 20 (2008),444-451.
13. T.A. Burton and S. Zhang, Unified boundedness, periodicity, and stability in ordinary and functional differential equations. Ann Mat. Pure Appl. 145 (1986), 129-158.
14. K. Gopalsamy, Stability and Oscillations in Delay Differential Equation of Population Dynamics. Kluwer Academic, Dordrecht (1992).
15. Cemil Tunc, Stability and boundedness of solutions of nonlinear differential equations Of third – order with delay,Electronic Journal, reg, NoP2375 at (1997).
16. S. Li and L. Wen, Functional Differential Equations. human Science and Technology press (1987).
17. A.U. Afuwape[I]; M.O.Omeike[II] Stability and boundedness of solutions of nonlinear differential equations of third – order with delay Comput. Appl. Math. Vol.29 no.3 Sao Carlos.
18. Y. Qin, Y. Liu and L. Wang, Stability of Motion for Dynamics Systems with Delay. Academic Press, Beijing (1966).

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