Research Article

Stability of solutions of nonlinear delay differential equations of third order

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Abstract In this paper we investigate stability of solutions of some nonlinear differential equations of third order with delay. By constructing a Lyapunov functional, sufficient conditions for the stability of solutions for equations considered are obtained.
Key Word: differential equations of third order.

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Email: choudharihl2011@gmail.com
Received Date: 08/12/2014 Accepted Date: 15/12/2014

Access this article online	
Quick Response Code:	Website: www.statperson.com
	DOI: 17 December 2014

INTRODUCTION

The Lyapunov function or functional approach has been a powerful tool to ascertain the stability of solutions of certain differential equations. Up to today, perhaps, the most effective method to determine the stability of solutions of non-linear differential equations is still the Lyapunov's direct (or second) method. The major advantage of this method is that the stability of solutions can be obtained without any prior knowledge of solutions.

For over four decades many authors made use of the Lyapunov's direct method by considering Lyapunov functional and obtained the conditions which ensure the stability of the problem, But how do we construct those appropriate Lyapuncv functional? No author has discussed them thus far. It is in general a difficult task. Similar problem is shared with ordinary differential equations of high orders[6].

We consider the third order non – autonomous nonlinear different equations with delays:

$$x''' + p(t)x'' + q(t)n(x') + m(x(t-r)) = 0$$

and

$$x''' + p(t)x'' + q(t)n(x') + m(x(t-r)) = P(t)$$

Where r is a positive constant, p(t), q(t),n(x), m(x) are real valued functions continuous in their respective arguments; n(0) = m(0)= 0. The dots indicates differential with respect to t and all solutions are assumed real.

Equations of the forms above equations in which p(t), q(t) are constants have been studied by several authors namely, Sadek [5] and Zhu[13], to mention a few . They obtained criteria which ensure the stability

How to site this article: Choudhari Hanumant Lahu, Kakde R V. Stability of solutions of nonlinear delay differential equations of third order. *International Journal of Statistika and Mathemtika* Nov 2014 to Jan 2015; 12(2): 120-125. http://www.statperson.com (accessed 18 December 2014).

Recently, in [6] , SADEK establishes conditions under which all solutions of the non- autonomus equation

$$x''' + p(t)x'' + q(t)n(x') + m(x(t-r) = 0$$

3.Preliminaries Now we will give the definitions and the stability criteria for the general non − autonmcus delay differentail system.

$$\dot{x} = f(t, x_t), x_t = x(t+\theta), -r \le \theta \le 0, t \ge 0 \quad -3.1$$

where $f: I \times C_H \to R^n$is a continuos mapping.

$$f(t,0) = 0, C_H = \{\emptyset \in (c[-r,0], R^n): ||\emptyset|| \le H$$

and forH_1, there exists $L(H_1) > 0$, *with*

$$|f|\emptyset| \le L(H_1) \text{when } ||\emptyset|| \le H_1.$$

Definition 3.1 : An element $\psi \epsilon C$is in the $\omega -$ limit set of $\emptyset$, say, $\Omega(\emptyset)$is defined on $[0,\infty)$and there is a sequence $\{t_n\}, t_n \to \infty$, as $n \to \infty$, with $\left||x_{tn}(\emptyset) - \psi|\right| \to 0$ as $n \to \infty$ where

$$x_{tn}(\emptyset) = x(t_n + \theta, 0, \emptyset) \text{for} - r \le \theta \le 0.$$

Definiton3.2:

A set $\subset C_H$is an invariant set if for any $\emptyset \epsilon Q$, the solution of (3.1), $x(t, 0, \emptyset)$isdefined on $[0,\infty)$and $x_t(\emptyset)\epsilon Q$for $t \epsilon [0,\infty)]$

Lemma 3.1. If $\emptyset \epsilon C_H$ is such that the solution $x_t(\emptyset)$of(3.1)with $x_0(\emptyset) = \emptyset$ is defined on $[0,\infty)$and $\left||x_t(\emptyset)|\right| \le H_1$for $t\epsilon[0,\infty)$, then$\Omega(\phi)$isa non − empty, compact, invariant set and

$$\text{dist}\left(x_t(\phi)\right), \Omega(\phi)) \to \infty \text{as } t \to \infty.]$$

Definiton: Stability and Boundedness of Solutions ofDelay Differential Equations third order

Lemma3.2.

Let $\vee(t,\emptyset): I \times C_H \to R$ be a continous functional satisfying a local Lipschitz condition $\vee(t, 0) = 0$, and such that:

i. $W_1(|\emptyset(0)|) \le \vee(t,\emptyset) \le W_2(||\emptyset||)$where $W_1(r), W_2(r)$, are wedges
ii. $\vee_{(3.1)}(t,\emptyset) \le 0$, for$\emptyset \epsilon C_H$.

Then the Zero solution of (3.1) is uniformly stable . If we define

$$\text{Then} Z = \left\{\emptyset \epsilon C_H : \vee_{(3.1)}(t,\emptyset) = 0\right\},$$

then the Zero solution (3.1) is asympotically stable, provided that the lagest invariant set in Z is $Q = \{0\}$

MAIN RESULTS

Theorem:

Suppose that $p(t), q(t) \epsilon C'(I), m\epsilon C'(R)$and $N\epsilon\, C(R)$& *these function Satisfy* following Condition

i. $m(0) = 0, \frac{m(x)}{x} \ge A_0 > 0, x \ne 0$
ii. $m'(x) \le C$
iii. $n(0) = 0, \frac{n(y)}{y} \ge q > 0, y \ne 0$
iv. $0 < A_1 \le q(t), -L \le q'(t) \le 0, t \epsilon I$
v. $0 < p \le p(t) \le L, t\epsilon I$

then euery Solation , $x = x(t)$ of

$$x''' + p(t)x'' + q(t)n(x') + m(x(t-r) = 0 \text{ is uniform bounded and Satisfies}$$

$x(t) \to 0, x'(t) \to 0, x''(t) \to 0\infty\ t \to \infty$ Praided there exist $\propto$ Satisfysing $\frac{q}{c} > \propto > \frac{1}{p}$ such that

vi. $\frac{1}{2}\ p'(t) \le A_2 < A_1(q - \propto c)\ t \epsilon I\ \&\ r < \min\left[\frac{2A_5}{3LC}, \frac{A_5}{L\propto C}\right]$

proof: − we write equation

$$x'(t) = y(t)$$
$$y'(t) = z(t)$$

$$z'(t) = -p(t)z - q(t)z - q(t)n(y) - m(x) + \int_{t-r}^{t} m'(x(s))y(s)\,ds$$

$\rightarrow (1)$

it's Lyapanou functional as

$$V\left(t, x(t), y(t), z(t)\right) = \int_0^x m(s)ds + \propto q(t)\int_0^y n(s)ds + \propto m(x)y$$
$$+\frac{1}{2}\,p(t)y^2 + Z \propto z^2 + yz + \mu_1 \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)\,d\,\theta\,ds + \mu_2 \int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\,ds$$

$\rightarrow (2)$

we, can also, assume that $0 < A_1 \le q(t) \le L$

$$\lim_{t\to\infty} q(t) = q_0$$
$$A_1 \le q_0 \le L$$

$\rightarrow (3)$

from (1) we write

$V\,(t, x(t), y(t), z(t)$

$$= \left[\int_0^x m(s)ds + \propto q(t)\int_0^y n(s)ds + \propto m(x)\,y\right]$$
$$+\frac{1}{2}[p(t)y^2 + yz + \propto z^2] + \mu_1 \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)\;d\theta + \mu_2 \int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\,ds$$
$$= V_1 + \frac{1}{2}\,V_2 + \mu_1 \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)\;d\theta + \mu_2 \int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\,ds$$

$\rightarrow (4)$

Now , we Consider,

$$V_2 = p(t)y^2 + yz + \propto z^2$$
$$= p(t)\left[y^2 + \frac{yz}{p(t)} + \frac{\propto z^2}{p(t)}\right]$$
$$= p(t)[y + \frac{z}{2\,p(t)}]^2 + \frac{1}{4\,p(t)}(4 \propto p(t) - 1)z^2$$

but $\propto p(t) \ge \propto p \ge 1$, since , $\propto > \frac{1}{p}$, $\propto$ is positive, $4 \propto p(t) - 1$ is positive thus, $A_3 > 0$

$$V_2 \ge \frac{1}{2}\,A_3\,y^2 + \frac{1}{2}\,A_3\,z^2 \;\rightarrow (5)$$

$$\text{Now,} V_1 = \left[\int_0^x m(s)ds + \propto q(t)\int_0^y n(s)ds + \propto m(x)y\right]$$

$$V_1 = [M(s) + \propto q(t)N(y) + \propto m(x)y]$$

where , $M(s) = \int_0^x m(s)ds$ & $N(y) = \int_0^y n(s)ds$.

$$V_1 \ge A_1\left[M(s) + \frac{\propto}{2}\,q\,y^2 + \propto m(s)y\right]$$

since, $q(t) \ge 1, A_1 > 0, \dfrac{n(y)}{y} \ge q > 0$

$$\Longrightarrow N(y) \geq \frac{1}{2} q\, y^2$$

$$\therefore\ \frac{1}{2}\ [2M(x)+\propto qy^2 +\ 2 \propto m(x)y]$$

$$= \frac{1}{2}\left[2M\,(x)+\propto q\, y^2 + 2 \propto m(x)\right] + \frac{\propto}{q} m^2(x) - \frac{\propto}{q} m^2(x)\Big]$$

$$= \frac{1}{2}\left[\frac{\propto}{q}\ (\ q^2\, y^2 +\ 2\, mqym(x) +\ m^2\,(x)\} +\ 2M(x) - \frac{\propto}{q}\ m^2\,(x)\right]$$

$$= \frac{1}{2}\,\frac{\propto}{q}\ (qy + m\,(x))^2 +\ \int_0^x [\,1 - \frac{\propto}{q} m'(s)]\ m(s)\ ds$$

$$\geq\ \int_0^x [\ 1 -\ \frac{\propto c}{q}]\ m(s)ds$$

$$\geq \int_0^x A_4 m(s)ds \text{ where } A_4 = 1 - \frac{\propto c}{q}$$

$$\geq\ A_4 \int_0^x m(s)ds$$

$$\geq\ A_4\ M(s)\ \rightarrow (6)$$

where, $A_4 =\ 1 - \frac{\propto c}{q} > 1 - \frac{\left(\frac{q}{c}\right) c}{b} = 0$

since , $\frac{m(x)}{x} > A_0 > 0$,we get $\vee \geq\ \frac{A_0 A_1 A_2}{2}\ x^2\ \rightarrow (7)$

adding (4), (5), & (7) we have .

$$\vee\left(t, x(t)\right), y(t), z(t))\ \geq\ \frac{A_0 A_1 A_2}{2}\ x^2 + \frac{A_3}{4}\ y^2 + \frac{A_3}{4}\ z^2 +\ \mu_1 \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)\ d\theta + \mu_2 \int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\ ds.$$

$$\vee\left(t, x(t)\right), y(t), z(t))\ \geq\ D_1 x^2 + D_2 y^2 + D_3 z^2 \geq\ D_4(x^2 + y^2 + z^2)$$

where,

$$D_4 = \min\{D_1,\ D_2, D_3\} \text{ since } \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)\ d\theta ds \geq 0\ \&\ \int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\ ds \geq 0$$

Hence ,

we check $\vee\left(t, x(t)\right), y(t), z(t))$ satisfies condition (I) of lemma 2.2.

from (1)& (2) we obtain.

$$\frac{d}{dt}\vee\,(t, x_t, y_t, z_t) = \propto q'(t)N(y) + \frac{1}{2}\ p'(t)y^2$$
$$- [q(t)\ n(y)y - \propto\ m'(x)y^2 -\ \mu r\ y^2]$$
$$- [\propto p(t) - 1\,]z^2 + y \int_{t-r}^{t} m'\left(x(s)\right)y(s)ds$$
$$+ \propto z \int_{t-r}^{t} m'\left(x(s)\right)\ y(s)\ \ ds -\ \mu\ \int_{t-r}^{t} y^2(s)ds\ \ \rightarrow (8)$$

Since, $m'(x)\ \leq c$, using $2|ab| \leq a^2 + b^2$

we obtain $\propto z \int_{t-r}^{t} m'\left(x(s)\right)\ y(s)\ \ ds\ \leq \frac{1}{2}\ \propto crz^2 + \frac{1}{2}\ c\ \int_{t-r}^{t} y^2(s)ds$

$\leq \frac{1}{2}\ L \propto crz^2 + \frac{1}{2}$

$$\&\ y \int_{t-r}^{t} m'\left(x(s)\right)y(s)ds\ \leq\ c \int_{t-r}^{t} y^2(s)ds$$

$$\leq Lc \int_{t-r}^{t} y^2(s)ds$$

$$\frac{d}{dt}\vee\,(t, x_t, y_t, z_t) \leq \propto q'(t)N(y) + \frac{1}{2}\ p'(t)y^2$$
$$- [q(t)\ n(y)y - \propto m'(x)y^2 - \mu\, ry^2]$$
$$- \frac{1}{2}\ [2(\propto (p(t) -\ Larc\,]z^2 + \left[\frac{3}{2}\ Lc - \mu\right] \int_{t-r}^{t} y^2(s)ds\ \ \rightarrow (9)$$

If $y = 0$ then $q(t), n(y)y - \propto m'(x)y^2 - \mu\, ry^2 = 0$

we can write

$$[q(t)\, n(y)y - \propto m'(x)y^2 - \mu\, ry^2] = \left[q(t)\frac{n(y)}{y} - \propto m'(x) - \mu\, r\right] y^2$$

$$\geq [q\, q(t) - \propto c - \mu r]y^2$$

$= [q(t)q - \propto c]y^2 - \mu ry^2$

$$\geq A_1[q - \propto c - A_1^{-1}\mu r]\, y^2$$

Thus,

$$\frac{1}{2}\, p'(t)y^2 - [q(t)\, n(y)y - \propto m'(x)y^2 - \mu\, ry^2]$$

$$\leq \left[\frac{1}{2} p'(t) - A_1(q - \propto c - A_1^{-1}\mu r\right]\, y^2$$

$$\leq\ [A_2 - A_1(q - \propto c) + \mu r]\, y^2$$

$\leq\ -[A_5 - \mu r]y^2\ \rightarrow (10)$

where, $A_5 = A_1[q - \propto c] - A_2 > 0\ by\ (4)$
accoding (5) ,$p(t) \geq \propto p > 1$

thus $[2(\propto (p(t) - 1) - L \propto rc\,]z^2 \geq (A_6 - L \propto cr)z^2\ \rightarrow (11)$ wher, $A_6 = \propto p - 1 > 0$
Substute(10), (11) into (9),also, $\mu = {}^3/_2\ LC$
we get, $\frac{d}{dt} \vee \left(t, x(t)\right), y(t), z(t)) \leq \propto q'(t) \int_0^y n(s)ds - \left(A_5 - \frac{3}{2}\ Lcr\right) y^2$

$$- (A_6 - L\ \propto cr)z^2\ \rightarrow (12)$$

Since , $q'(t) \leq 0$ & $\int_0^y n(s)ds\ \geq 0$,$\propto\ q'(t) \int_0^y n(s)ds\ \leq 0$
since in (6) ,$M(x) + \propto N(y) + \propto m(x)y \geq\ A_4 M(x) \geq 0$
Thus,

$\frac{d}{dt} \vee \left(t, x(t)\right), y(t), z(t)) \leq\ -(\, A_5 - \frac{3}{2}\ Lcr)y^2\ -\ (A_6 - L\ \propto cr)z^2$

if $r < \min\left[\frac{2A_5}{3LC}, \frac{A_5}{L \propto C}\right]$

we have, $\frac{d}{dt} \vee \left(t, x(t)\right), y(t), z(t)) \leq\ -\beta\, (y^2 + z^2), \beta > 0$

by $\frac{d}{dt} \vee \left(t, x(t)\right), y(t), z(t)) = 0$

system (1)we can easily obtain
$x = y = z = 0$, Condition lemma (3.2)are satisfied
∴ ptoof is conplete.

CONCLUSIONS

In this paper, we considered the third order non-autonomous non-linear differential equations with delays. The differential equations we have discussed in this paper in which p (t), q (t) are constants have been studied by several authors. They obtained criteria which ensure the stability of solutions. Sadek establishes conditions under which all solutions of the non-autonomous equation tend to the zero solution as $t \rightarrow \infty$. These results are now extended by considering the semi-invariant set of a related non-autonomous system.

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Source of Support: None Declared
Conflict of Interest: None Declared