Research Article

Almost sure limit points of maxima of stationary gaussian sequences through at least geometrically fast subsequences

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Abstract Let (X_n, n≥1) be a discrete - parameter Stationary Gaussian process with E (X_i)=0, EX_i^2=1forall I and $E(X_i\ X_{i+n})$=r(n). Let (Y_n, n≥1) be an independent copy of (X_n, n≥1). Let M_{1n}= max $(X_1, X_2,\ldots,X_n)$, M_{2n}= max$(Y_1, Y_2,\ldots,Y_n)$, $U_n=\frac{(M_{1n}-b_n)}{a_n}$ and $V_n=\frac{(M_{2n}-b_n)}{a_n}$ where $b_n= (2\log n)^{1/2}$ and a_n=(log log n)$(2\log n)^{-1/2}$. Let (n_k) be a subsequence of positive integers that is at least geometrically fast. Under the condition that either $(\log n)^{1+\gamma}$r(n)=O(1) as n→ ∞ for some γ >0 or $\sum_{j=1}^{\infty} r^2$(j) < ∞ the set of all almost sure limit points of the vector sequenc e $\left\{ U_n = \frac{(M_{1n}-b_n)}{a_n}, V_n = \frac{(M_{2n}-b_n)}{a_n} \right\}$ is obtained.

Keywords: stationary gaussian sequence.

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Received Date: 28/01/2015 Revised Date:03/02/2015 Accepted Date: 06/02/2015

Access this article online	
Quick Response Code:	Website: www.statperson.com
	DOI: 08 February 2015

INTRODUCTION

Let (X_n, n≥1) be a discrete - parameter Stationary Gaussian proc ess with E(X_i)=0, EX_i^2 =1 for all I and $E(X_i\ X_{i+n})$=r(n). Let (Y_n, n≥1) be an independent copy of (X_n, n≥1). Let M_{1n}= max $(X_1, X_2,\ldots,X_n)$, M_{2n}= max $(Y_1, Y_2,\ldots,Y_n)$, $U_n=\frac{(M_{1n}-b_n)}{a_n}$ and $V_n=\frac{(M_{2n}-b_n)}{a_n}$ where $b_n= (2\log n)^{1/2}$ and a_n=(log log n)$(2\log n)^{-1/2}$. Pickands (1969) established that if either $(\log n)^{\alpha}$r(n) → 0 as n→ ∞ for some α >0 or $\sum_{j=1}^{\infty} r^2$(j) < ∞ then almost surely, as n→ ∞ lim sup $U_n = \frac{1}{2}$ and lim inf U_n= - $\frac{1}{2}$. Mittal (1974) showed that the above results continue to hold if the condition $(\log n)^{\alpha}$r(n) → 0 as n→ ∞ for some α >0 is replaced by the weaker condition $(\log n)^{\alpha}$r(n) =O(1) as n→ ∞ for some α >0. Vishnu Hebbar (1980) obtained the almost sure limit set of (U_n, V_n) when either $(\log n)^{1+\gamma}$r (n) =O (1) as n→ ∞ for some γ >0 or $\sum_{j=1}^{\infty} r^2$(j) < ∞. Let (n_k) be a subsequence of positive integers with $n_k \to \infty$

How to site this article: S S Nayak, Varalaxmi T Shedole. Almost sure limit points of maxima of stationary gaussian sequences through at least geometrically fast subsequences. *International Journal of Statistika and Mathemtika* Feb. to Apr. 2015; 13(1): 08-14.
http://www.statperson.com (accessed 10 February 2015).

as k $\rightarrow \infty$. It is said to be at least geometrically fast if lim sup $\left(\frac{n_k}{n_{k+1}}\right)$< 1. Vasudeva and Savitha (1995) established the law of the iterated logarithm for $\left(U_{n_k}\right)$. In this paper, we extend their result to the vector case by finding the almost sure limit set of $\left(U_{n_k}, V_{n_k}\right)$. Almost sure limit points of random vectors with independent components have received considerable attention in literature. One can refer the papers by LePage (1973), Pakshirajan and Vasudeva (1977), Strassen (1964), Nayak,S.S. (1984, 1985, 1986, 1988,1994,2000 and 2001) and the references therein. Throughout the paper, const. stands for a positive constant which may have different values at different appearances. "infinitely often" is written as i.o.

PRELIMINARIES

Let (n_k) be a subsequence of positive integers such that $n_k \rightarrow \infty$ as k $\rightarrow \infty$lim and
lim sup $\left(\frac{n_k}{n_{k+1}}\right)$< 1.
Let ε^*= inf$\left\{\varepsilon\text{: } \sum(\log n_k)^{-(\varepsilon+1/2)} < \infty\right\}$. For -$\frac{1}{2}$<$x_i \leq \varepsilon^*$, i=1,2 with -1< x_1+$x_2 \leq \varepsilon^*$-$\frac{1}{2}$, let n_k^*=$n_{u(k)}$ where u(k)= $\left[k^{\frac{1}{1+x_1+x_2}}\right]$ and $[x]$ is the greatest integer $\leq x$. Let m_k= $\left[n_k^*(\log k)^{-\frac{1}{2}}\right]$. Let us define the events F_k =$\left\{\max_{n_k^*-m_k+1\leq j\leq n_k^*} X_j > d_{n_k^*}(x_1)\right\}$ and G_k =$\left\{\max_{n_k^*-m_k+1\leq j\leq n_k^*} Y_j > d_{n_k^*}(x_2)\right\}$ where
$d_{n_k^*}(x)$ =$a_{n_k^*}$x+$b_{n_k^*}$. When r(n)=0, the corresponding events are respectively denoted by F_k^* and G_k^*.
Lemma 2.1 (Vasudeva and Savitha, 1995): Assume that either $(\log n)^{1+\gamma}$r(n)=O(1) as n→ ∞ for some γ >0 or $\sum_{j=1}^{\infty} r^2$(j) $< \infty$. Then lim sup U_{n_k}=ε^* a.s.
Lemma 2.2 (Vasudeva and Savitha, 1995): Assume that either $(\log n)^{1+\gamma}$r(n)=O(1) as n→ ∞ for some γ >0 or $\sum_{j=1}^{\infty} r^2$(j) $< \infty$. Then for any subsequence (v_k) of positive integers with $\lim_{k\rightarrow\infty} v_k$=∞ we have lim inf U_{v_k}= -$\frac{1}{2}$.
Lemma 2.3: Let 0<α<$\frac{1-\delta}{1+\delta}$ and
$A_1(k)$=$\sum_{j=1}^{[m_k^\alpha]} |r(j)|\,(m_k - j)(1 - r^2(j))^{-1/2}exp\left\{-\frac{d^2_{n_k^*}(x_1)}{(1+|r(j)|)}\right\}$ where $sup_{n\geq1}|r(n)|$=δ $(0 < \delta < 1)$.
Then $\sum_{k=1}^{\infty} P(G_k^*) A_1$(k) <∞ .
Proof: Stationarity implies $sup_{n\geq1}|r(n)|$=δ $(0 < \delta < 1)$. We have
$A_1(k)$ < (const.)$(n_k^*)^{-\frac{2}{1+\delta}}$ $(\log n_k^*)^{-\frac{2x_1}{1+\delta}}$ $\sum_{j=1}^{[m_k^\alpha]}(m_k - j)exp\left\{-\frac{1}{1+\delta}(2\log n_k^* + 2x_1 \log\log n_k^*) + o(1)\right\}$
<(const.) ${m_k}^{\alpha+1}(n_k^*)^{-\frac{2}{1+\delta}}(\log n_k^*)^{-\frac{2x_1}{1+\delta}}$, k≥ k_1
<(const.)$(\log k)^{-\frac{\alpha+1}{2}}$ $(\log n_k^*)^{-\frac{2x_1}{1+\delta}}(n_k^*)^{\alpha+1-\frac{2}{1+\delta}}$, k≥ k_1.
Also, P(G_k^*)=1 - $\emptyset^{m_k}\left(d_{n_k^*(x_2)}\right)$ where $\emptyset$ is the d.f. of a standard normal random variable.
= 1- exp$\left\{m_k \log\left\{1 - \left(1 - \emptyset\left(d_{n_k^*(x_2)}\right)\right)\right\}\right\}$
= 1- exp$\left\{-m_k\left(1 - \emptyset\left(d_{n_k^*(x_2)}\right)\right)(1 + o(1))\right\}$
<(const.)$(\log k)^{-1/2}$ $(\log n_k^*)^{-(x_2+1/2)}$ k≥ k_2 by the tail behavior of the standard normal distribution.
Let N=max(k_1, k_2). Then
$\sum_{k=N}^{\infty} P(G_k^*) A_1(k)$ < (const.)$\sum_{k=N}^{\infty}(\log k)^{-\frac{\alpha+2}{2}}$ $(n_k^*)^{-\theta}$ $(\log n_k^*)^{-\beta}$ where $\theta = \frac{2}{1+\delta} - \alpha - 1$ >0
and $\beta = \frac{2x_1}{1+\delta}$+$x_2 + \frac{1}{2}$.
< (const.) $\sum_{k=N}^{\infty}(n_k^*)^{-\theta}$ $(\log n_k^*)^{-\beta}$
= (const.) $\sum_{k=N}^{\infty} e^{-\theta \log n_k^*}$ $(\log n_k^*)^{-\beta}$
<(const.) $\sum_{k=N}^{\infty}$ $(\log n_k^*)^{-(l+\beta)}$ where l > max$(1, 1 + x_1 + x_2 - \beta)$ is an integer.
Since lim sup$\frac{n_k}{n_{k+1}}$ < 1, we have n_k^*> (const.)$a^{u(k)}$ for k $\geq k_3$ where a>1 and u(k)= $\left[k^{\frac{1}{1+x_1+x_2}}\right]$. Hence $\sum_{k=N}^{\infty} P(G_k^*) A_1(k)$ < (const.)+ (const.)$\sum_{k=k_3}^{\infty} k^{-\frac{l+\beta}{1+x_1+x_2}}$ <∞.

Lemma 2.4: Let $0<\alpha<\frac{1-\delta}{1+\delta}$ where $sup_{n\geq 1}|r(n)|=\delta\ (0<\delta<1)$. Let $sup_{k\geq n}|r(k)|=\delta(n)$ and $A_2(k)=\sum_{j=[m_k^\alpha]+1}^{m_k-1}|r(j)|\,(m_k-j)(1-r^2(j))^{-1/2}\exp\left\{-\frac{d_{n_k^*}^2(x_1)}{(1+|r(j)|)}\right\}$. Assume that either $(\log n)^{1+\gamma}$r(n)=O(1) as n→ ∞ for some $\gamma>0$ or $\sum_{j=1}^\infty r^2(j)<\infty$. Then $\sum_{k=1}^\infty P(G_k^*)A_2(k)<\infty$.

Proof: First let $(\log n)^{1+\gamma}$r(n)=O(1) as n→ ∞ for some $\gamma>0$. Then $(\log n)^{1+\gamma}\,\delta(n)$=O(1) as n→ ∞ for some $\gamma>0$. Now

$A_2(k)$ < (const.)$\delta(m_k^\alpha)(1-\delta^2)^{-1/2}(n_k^*)^{-\frac{2}{1+\delta(m_k^\alpha)}}(\log n_k^*)^{-\frac{2}{1+\delta(m_k^\alpha)}}\sum_{j=[m_k^\alpha]+1}^{m_k-1}(m_k\text{-j})$,k≥ k_4

< (const.) $\delta(m_k^\alpha)\, m_k^2\,(n_k^*)^{-\frac{2}{1+\delta(m_k^\alpha)}}(\log n_k^*)^{-\frac{2x_1}{1+\delta(m_k^\alpha)}}$, k≥ k_5

< (const.) $\delta(m_k^\alpha)\, m_k^{2\delta(m_k^\alpha)}(\log k)^{\delta(m_k^\alpha)}(\log n_k^*)^{-\frac{2x_1}{1+\delta(m_k^\alpha)}}$, k≥ k_6 since $m_k\sim n_k^*(\log k)^{-1/2}$as k→ ∞.

=(const.) $\delta(m_k^\alpha)\,(\log k)^{\delta(m_k^\alpha)}(\log n_k^*)^{-\frac{2x_1}{1+\delta(m_k^\alpha)}}\exp\{2\delta(m_k^\alpha)\log m_k\}$, k≥ k_6

<(const.) $(\log m_k)^{-\gamma-1}(\log k)^{\delta(m_k^\alpha)}(\log n_k^*)^{-\frac{2x_1}{1+\delta(m_k^\alpha)}}$, k≥ k_7 since $\delta(m_k^\alpha)(\log m_k^\alpha)^{1+\gamma}$ =O(1) as k→ ∞.

Since lim sup$\frac{n_k}{n_{k+1}}<1$, we have $n_k^*>$ (const.)$a^{u(k)}$ for k large where a>1 and u(k)= $\left[k^{\frac{1}{1+x_1+x_2}}\right]$. This implies that $\frac{\log\log k}{\log n_k^*}$ $\to 0\ as\ k\to\infty$. Hence $\log m_k\sim\log n_k^*\ as\ k\to\infty$. Hence

$A_2(k)$ <(const.) $(\log k)^{\delta(m_k^\alpha)}(\log n_k^*)^{-\frac{2x_1}{1+\delta(m_k^\alpha)}-\gamma-1}$, k≥ k_8.

Also, P(G_k^*)<(const.)$(\log k)^{-1/2}\,(\log n_k^*)^{-(x_2+1/2)}$ for large k by the tail behavior of the standard normal distribution. Hence

P(G_k^*)A_2(k)<(const.) $(\log k)^{\delta(m_k^\alpha)-1/2}(\log n_k^*)^{-\frac{2x_1}{1+\delta(m_k^\alpha)}-x_2-\gamma-3/2}$, k ≥ k_9.

<(const.) $(\log k)^{\delta(m_k^\alpha)-1/2}k^{-\left(\frac{2x_1}{1+\delta(m_k^\alpha)}+x_2+\gamma+3/2\right)/(1+x_1+x_2)}$, k ≥ k_{10} since lim sup$\frac{n_k}{n_{k+1}}<1$ implies $n_k^*>$ (const.)$a^{u(k)}$ for k large where a>1 and u(k)= $\left[k^{\frac{1}{1+x_1+x_2}}\right]$.

Note that $(\log n)^{1+\gamma}\ \delta(n)$ =O(1) as n→ ∞ for some $\gamma>0$ implies $\delta(m_k^\alpha)\to 0$ as k→ ∞.

Let $0<\varepsilon_1<\frac{x_1+\gamma+1/2}{1+x_1+x_2}$. This is possible since $x_1>-\frac{1}{2}$ and $1+x_1+x_2>0$. Hence

$\sum_{k=1}^\infty P(G_k^*)A_2$(k) < const. +(const.)$\sum_{k=k_{11}}^\infty(\log k)^{\varepsilon_1-1/2}\ k^{\left(\varepsilon_1-\frac{2x_1+x_2+\gamma+3/2}{1+x_1+x_2}\right)}<\infty$.

Now let $\sum_{j=1}^\infty r^2$(j) < ∞. By Cauchy-Schwarz inequality we have

$(A_2(k))^2<\sum_{j=[m_k^\alpha]+1}^{m_k-1}r^2(j)\sum_{j=[m_k^\alpha]+1}^{m_k-1}(m_k-j)^2(1-r^2(j))$ X

$\exp\left\{-\frac{2}{1+|r(j)|}\left\{2\log n_k^*+2x_1\log\log n_k^*+{x_1}^2\frac{(\log\log n_k^*)^2}{2\log\log n_k^*}\right\}\right\}$

<(const.)$\left\{\sum_{j=1}^\infty r^2(j)\right\}(n_k^*)^{-\frac{4}{1+\delta(m_k^\alpha)}}(\log n_k^*)^{-\frac{4x_1}{1+\delta(m_k^\alpha)}}\sum_{j=[m_k^\alpha]+1}^{m_k-1}(m_k^*-j)^2$, k ≥ k_{11}

< (const.)$\left\{\sum_{j=1}^\infty r^2(j)\right\}(n_k^*)^{-\frac{4}{1+\delta(m_k^\alpha)}}(\log n_k^*)^{-\frac{4x_1}{1+\delta(m_k^\alpha)}}\,m_k^3$, k ≥ $k_{12.}$

< (const.)$\left\{\sum_{j=1}^\infty r^2(j)\right\}(n_k^*)^{3-\frac{4}{1+\delta(m_k^\alpha)}}(\log k)^{-3/2}(\log n_k^*)^{-\frac{4x_1}{1+\delta(m_k^\alpha)}}$, k ≥ k_{13} since m_k=$\left[n_k^*(\log k)^{-\frac{1}{2}}\right]$.

Hence A_2(k) < (const.)$\left\{\sum_{j=1}^\infty r^2(j)\right\}^{1/2}(n_k^*)^{3/2-\frac{2}{1+\delta(m_k^\alpha)}}(\log k)^{-3/4}(\log n_k^*)^{-\frac{2x_1}{1+\delta(m_k^\alpha)}}$, k ≥ k_{13}

<(const.)$\left\{\sum_{j=1}^\infty r^2(j)\right\}^{1/2}(n_k^*)^{\epsilon_1-1/2}(\log k)^{-3/4}(\log n_k^*)^{\epsilon_1-2x_1}$, k ≥ k_{14} ,0<$\epsilon_1<\frac{1}{2}$,
since $\delta(m_k^\alpha)\to 0\ as\ k\to\infty$.

<(const.)$\left\{\sum_{j=1}^\infty r^2(j)\right\}^{1/2}(\log n_k^*)^{\epsilon_1-2x_1}\exp\{-\theta\log n_k^*\}$, k ≥ k_{15} where $\theta=\frac{1}{2}-\epsilon_1$.

<(const.)$\left\{\sum_{j=1}^\infty r^2(j)\right\}^{1/2}(\log n_k^*)^{\epsilon_1-2x_1-l}$, k ≥ k_{15} where l is an integer such that
l ≥ $max(1,\varepsilon^*+\varepsilon_1-2x_1-x_2)$.

Also P(G_k^*)<(const.)$(\log k)^{-1/2}$ $(\log n_k^*)^{-(x_2+1/2)}$ for large k by the tail behavior of the standard normal distribution. Hence,

$\sum_{k=1}^{\infty} P(G_k^*)A_2$(k)< (const.) + (const.) $\left\{\sum_{j=1}^{\infty} r^2(j)\right\}^{1/2}$ $\sum_{k=k_{16}}^{\infty}$ $(\log n_k^*)^{\epsilon_1-2x_1-l-x_2-1/2}$ <∞.

Lemma 2.5: Let I_k be the indicator function of the event E_k. Assume that $(\log n)^{1+\gamma}$r(n)=O(1) as n→ ∞ for some γ >0. Then $\sup_n|\sum\sum_{N\le k<l\le n} Cov(I_k, I_l)|$<∞ where N is a large positive integer.

Proof: We have for k<l,

$Cov(I_k, I_l)$ =E(I_kI_l) – E(I_k) E(I_l)

=P($F_k \cap F_l$) P($G_k \cap G_l$) - P(F_k) P(F_l) P(G_k) P(G_l)

={P($F_k \cap F_l$) − P(F_k)P(F_l)}({P($G_k \cap G_l$) − P(G_k)P(G_l)} + P(G_k)P(G_l))

+ P(F_k) P(F_l) {P($G_k \cap G_l$) − P(G_k) P(G_l)}. Therefore

$|Cov(I_k, I_l)|$ ≤ {|P($F_k \cap F_l$) − P(F_k)P(F_l)|}{|P($G_k \cap G_l$) − P(G_k)P(G_l)|}

+ P(G_k)P(G_l){|P($F_k \cap F_l$) − P(F_k)P(F_l)|} + P(F_k) P(F_l){|P($G_k \cap G_l$) − P(G_k)P(G_l)|}. (2.1)

By Qualls and Watanabe (1971) we get

|P($F_k \cap F_l$) − P(F_k)P(F_l)|=|P($F_k^c \cap F_l^c$) − P(F_k^c)P(F_l^c)| (2.2)

$\le \sum_{\mu=1}^{m_l}\sum_{v=1}^{m_k}|r|\int_0^1 \varphi\left(d_{n_k^*}(x_1), d_{n_l^*}(x_1): \lambda r\right)d\lambda$ where $\varphi(u, v, \rho)$ is the standard bivariate normal density with correlation coefficient ρ and r=r($n_l^* - m_l + \mu - n_k^* + m_k - v$). Since $(\log n)^{1+\gamma}$r(n)=O(1) as n→ ∞, we have $(\log n)^{1+\gamma}$ $\delta(n)$=O(1) as n→ ∞ where $sup_{k\ge n}|r(k)|=\delta(n)$. Stationarity of X_n's and the condition on r(n) ensure that $\delta(1) < 1$.

Let lim sup $\frac{n_k^*}{n_{k+1}^*}$= a(0<a<1). Hence, noting that $\frac{m_l}{n_l^*} \to 0$ as $l \to \infty$, we have,

$n_l^* - m_l + \mu - n_k^* + m_k - v \ge (const.)n_l^*$ for l large.

Therefore$|n_l^* - m_l + \mu - n_k^* + m_k - v| \le \delta((const.)n_l^*) \le (const.)$ $(\log n_l^*)^{-1-\gamma}$ for large k and l such that l>k≥ N, N being a sufficiently large positive integer.

Note that

$\varphi\left(d_{n_k^*}(x_1), d_{n_l^*}(x_1): \lambda r\right) \le (2\pi)^{-1}\left((1-\delta(1))\right)^{-1/2} \times$

$\exp\left\{-\frac{1}{2}\left(d_{n_k^*}{}^2(x_1) - 2|r|d_{n_k^*}(x_1)d_{n_l^*}(x_1) + d_{n_l^*}{}^2(x_1)\right)\right\}$ (2.3)

≤ (const.) $\exp\left\{-\frac{1}{2}\left(d_{n_k^*}{}^2(x_1) + (1-2|r|)d_{n_l^*}{}^2(x_1)\right)\right\}$ because $d_{n_j^*}$'s are monotonically increasing in j.

< (const.)$(n_k^*)^{-1}(\log n_k^*)^{-x_1}(n_l^*(\log n_l^*)^{x_1})^{-\left(1-2\delta((const.)n_l^*)\right)}$

=(const.)$(n_k^*)^{-1}(\log n_k^*)^{-x_1}(n_l^*)^{-1}(\log n_l^*)^{-x_1}$ $\exp\{2\delta((const.)n_l^*)(\log n_l^* + x_1\log\log n_l^*)\}$

< (const.)$(n_k^*)^{-1}(\log n_k^*)^{-x_1}(n_l^*)^{-1}(\log n_l^*)^{-x_1}$ $\exp\{2(const.)(\log n_l^*)^{-1-\gamma}(\log n_l^* + x_1\log\log n_l^*)\}$

since $(\log n)^{1+\gamma}$ $\delta(n)$=O(1) as n→ ∞.

< (const.)$(n_k^*)^{-1}(\log n_k^*)^{-x_1}(n_l^*)^{-1}(\log n_l^*)^{-x_1}$.

Hence the R.H.S of (2.2) can be majorized by

(const.) m_km_l $\delta((const.)n_l^*)(n_k^*)^{-1}(\log n_k^*)^{-x_1}(n_l^*)^{-1}(\log n_l^*)^{-x_1}$ (2.4)

≤ (const.)$(\log k)^{-1/2}(\log l)^{-1/2}(\log n_k^*)^{-x_1}(\log n_l^*)^{-x_1-\gamma-1}$

Similarly |P($G_k \cap G_l$) − P(G_k)P(G_l)| can be majorized by an expression which is obtained from (2.4) By replacing x_1 by x_2 . Hence the first term of R.H.S of (2.1) is

≤ (const.)$(\log k)^{-1}(\log l)^{-1}(\log n_k^*)^{-(x_1+x_2)}(\log n_l^*)^{-(x_1+x_2+2\gamma+2)}$

≤ (const.)$(\log k)^{-1}(\log n_k^*)^{-(x_1+x_2+\gamma+1)}(\log l)^{-1}(\log n_l^*)^{-(x_1+x_2+\gamma+1)}$.

The second term of (2.1) for large l> $k \ge N$ large is

$\le m_k$P$\left(Y_1 > d_{n_k^*}(x_2)\right)m_l$ P$\left(Y_1 > d_{n_l^*}(x_2)\right)$ {|P($F_k \cap F_l$) − P(F_k)P(F_l)|}

≤(const.) $m_km_l(\log k)^{-3/2}$ $(\log l)^{-1}(\log n_k^*)^{-(x_1+x_2+1/2)}(\log n_l^*)^{-(x_1+x_2+\gamma+3/2)}$

≤(const.) $(\log k)^{-1}(\log n_k^*)^{-(x_1+x_2-1/2)}(\log n_l^*)^{-(x_1+x_2+\gamma+3/2)}$

≤(const.)$(\log k)^{-1}(\log n_k^*)^{-(x_1+x_2+\gamma/2+1)}(\log l)^{-1}(\log n_l^*)^{-(x_1+x_2+\gamma/2+1)}$.

The third term of (2.1) is bounded by the same expression as the second. From these bounds the proof the lemma is complete.

Lemma 2.6: Let I_k be the indicator function of the event E_k. Assume that $\sum_{j=1}^{\infty} r^2$(j) < ∞. Then $\sup_n|\sum\sum_{N\le k<l\le n} Cov(I_k, I_l)|$<∞ where N is a large positive integer.

Proof: We have

$|Cov(I_k, I_l)| \leq |\mathrm{P}(G_k \cap G_l) - \mathrm{P}(G_k)\mathrm{P}(G_l)| + |\mathrm{P}(F_k \cap F_l) - \mathrm{P}(F_k)\mathrm{P}(F_l)|$

$$\leq \sum_{\mu=1}^{m_l} \sum_{\upsilon=1}^{m_k} |r| \left(\int_0^1 \varphi\left(d_{n_k^*}(x_1), d_{n_l^*}(x_1) : \lambda r\right) \mathrm{d}\lambda + \int_0^1 \varphi\left(d_{n_k^*}(x_2), d_{n_l^*}(x_2) : \lambda r\right) \mathrm{d}\lambda \right)$$

by lemma 1.5 of Qualls and Watanabe(1971) where r is defined at (2.2) of the proof of lemma 2.5.

The above expression can be majorized by

$$\sum_{\mu=1}^{m_l} \sum_{\upsilon=1}^{m_k} |r| \left(\exp\left\{-\frac{1}{2}\left(d_{n_k^*}{}^2(x_1) + (1-2|r|) d_{n_l^*}{}^2(x_1)\right)\right\} + \exp\left\{-\frac{1}{2}\left(d_{n_k^*}{}^2(x_2) + (1-2|r|) d_{n_l^*}{}^2(x_2)\right)\right\} \right) \quad (2.5)$$

(c.f. (2.3) of the proof of lemma 2.5)

As in (2.2) of the proof of lemma 2.5, we have $n_l^* - m_l + \mu - n_k^* + m_k - \nu \geq (const.) n_l^*$ for l large.

Since r(n)→ 0 as n→ ∞, we get

$|r| = |r(n_l^* - m_l + \mu - n_k^* + m_k - \nu)| < \varepsilon$ for l >k $\geq N$ where ε>0 is sufficiently small. Hence (2.5) can be majorized by

$$\left(\exp\left\{-\frac{1}{2}\left(d_{n_k^*}{}^2(x_1) + (1-2\,\varepsilon) d_{n_l^*}{}^2(x_1)\right)\right\} + \exp\left\{-\frac{1}{2}\left(d_{n_k^*}{}^2(x_2) + (1-2\,\varepsilon) d_{n_l^*}{}^2(x_2)\right)\right\} \right) \times$$

$$\sum_{\mu=1}^{m_l} \sum_{\upsilon=1}^{m_k} |r(n_l^* - m_l + \mu - n_k^* + m_k - \nu)| \quad (2.6)$$

By Cauchy-Schwarz inequality

$$\sum_{\mu=1}^{m_l} \sum_{\upsilon=1}^{m_k} |r(n_l^* - m_l + \mu - n_k^* + m_k - \nu)|$$

$$\leq m_l^{1/2} \left(\sum_{\mu=1}^{m_l} \left(\sum_{\upsilon=1}^{m_k} |r(n_l^* - m_l + \mu - n_k^* + m_k - \nu)| \right)^2 \right)^{1/2}$$

$$\leq m_l^{1/2} m_k^{1/2} \left(\sum_{\mu=1}^{m_l} \sum_{\upsilon=1}^{m_k} r^2 (n_l^* - m_l + \mu - n_k^* + m_k - \nu) \right)^{1/2}$$

$\leq m_l^{\frac{1}{2}} m_k \left(\sum_{j=1}^{\infty} r^2(\mathrm{j}) \right)^{\frac{1}{2}}$ since

$$\sum_{\mu=1}^{m_l} \sum_{\upsilon=1}^{m_k} r^2 (n_l^* - m_l + \mu - n_k^* + m_k - \nu)$$

$$= \sum_{\mu=1}^{m_l} \left(r^2(n_l^* - m_l + \mu - n_k^* + m_k - 1) + \cdots \ldots \ldots + r^2(n_l^* - m_l + \mu - n_k^*) \right)$$

$$\leq m_k \left(\sum_{j=1}^{\infty} r^2(\mathrm{j}) \right)^{\frac{1}{2}}.$$

Thus (2.6) is bounded by

$$(\text{const.})\, m_l^{\frac{1}{2}} m_k (n_k^*)^{-1} (n_l^*)^{-(1-2\varepsilon)} \left((\log n_k^*)^{-x_2} (\log n_l^*)^{-(1-2\varepsilon)x_2} + (\log n_k^*)^{-x_1} (\log n_l^*)^{-(1-2\varepsilon)x_1} \right)$$

This complete the proof of the lemma.

THE MAIN RESULT

Theorem 3.1: Assume that either $(\log n)^{1+\gamma}$r(n)=O(1) as n→ ∞ for some γ >0 or $\sum_{j=1}^{\infty} r^2(\mathrm{j}) < \infty$. Let (n_k) be any subsequence of positive integers such that $n_k \to \infty$as k→ ∞ and lim sup $\frac{n_k}{n_{k+1}}$<1. Then the set of all almost sure limit points of (U_{n_k}, V_{n_k}) is

$S_2 = \left\{ (x_1, x_2) : -\frac{1}{2} \leq x_1, x_2 \leq \frac{1}{2}, x_1 + x_2 \leq \varepsilon^* - \frac{1}{2} \right\}$ where $\varepsilon^* = \inf\{\varepsilon : \sum (\log n_k)^{-(\varepsilon+1/2)} < \infty\}$.

The proof of the theorem is based on the following three lemmas. Let n_k^*, m_k , F_k and G_k be as in section2.

Lemma 3.1: For all $x_1, x_2 > -\frac{1}{2}$ and for all $\varepsilon > 0$, we have

(i) $\mathrm{P}\left(U_{n_k^*} > x_1 + \epsilon, V_{n_k^*} > x_2\ i.o.\right)$ =0 and (ii) $\mathrm{P}\left(U_{n_k^*} > x_1, V_{n_k^*} > x_2 + \epsilon\ i.o.\right)$ =0

Proof: Since $U_{n_k^*}$ and $V_{n_k^*}$ are independent,

$\mathrm{P}\left(U_{n_k^*} > x_1 + \epsilon, V_{n_k^*} > x_2\right) = \mathrm{P}\left(U_{n_k^*} > x_1 + \epsilon\right) \mathrm{P}\left(V_{n_k^*} > x_2\right)$

$\leq n_k^{*2} \mathrm{P}\left(X_1 > d_{n_k^*}(x_1 + \epsilon)\right) \mathrm{P}\left(Y_1 > d_{n_k^*}(x_2)\right)$

$\sim (const.) (\log n_k^*)^{-(1+x_1+x_2+\varepsilon)}$ as k→ ∞ , using the known result

$1\text{-}\emptyset(x) \sim (2\pi)^{-1/2}x^{-1}\exp\left(-\frac{x^2}{2}\right)$ as x→ ∞ for the standard normal distribution function $\emptyset$. Since lim sup $\frac{n_k}{n_{k+1}} < 1$, we have $n_k > a^k$, 0<a<1 for k large. Hence $\log n_k^* >$ (const.)u(k) for k≥ k_0 where u(k)= $\left[k^{\frac{1}{1+x_1+x_2}}\right]$ and $[x]$ is the greatest integer $\leq x$. Thus $\sum_k \mathrm{P}\left(U_{n_k^*} > x_1+\epsilon, V_{n_k^*} > x_2\right) < \infty$. An application of the Borel-Cantelli lemma completes the proof of (i). Proof of (ii) is similar.

Lemma 3.2: Assume that either $(\log n)^{1+\gamma}$r(n)=O(1) as n→ ∞ for some γ >0 or $\sum_{j=1}^{\infty} r^2(j) < \infty$.
For all $x_1, x_2 > -\frac{1}{2}$ with $x_1 + x_2 \leq \varepsilon^* - \frac{1}{2}$, we have
$\mathrm{P}\left(U_{n_k^*} > x_1, V_{n_k^*} > x_2\ i.o.\right) = 1$.

Proof: Recall that let $n_k^* = n_{u(k)}$ where u(k)= $\left[k^{\frac{1}{1+x_1+x_2}}\right]$ and $[x]$ is the greatest integer $\leq x$, $m_k = \left[n_k^*(\log k)^{-\frac{1}{2}}\right]$, $F_k = \left\{\max_{n_k^*-m_k+1\leq j\leq n_k^*} X_j > d_{n_k^*}(x_1)\right\}$and,
$G_k = \left\{\max_{n_k^*-m_k+1\leq j\leq n_k^*} Y_j > d_{n_k^*}(x_2)\right\}$ where $d_{n_k^*}(x) = a_{n_k^*}x + b_{n_k^*}$. When r(n)=0, the corresponding events are respectively denoted by F_k^* and G_k^*. Define $E_k = F_k \cap G_k$. Observe that $E_k \subset \left\{U_{n_k^*} > x_1, V_{n_k^*} > x_2\right\}$. Hence the lemma will be established if we show that

$$\mathrm{P}(E_k \text{ i.o.})=1. \qquad (3.1)$$

This in turn will follow when we show as n→ ∞ that

$$\mathrm{E}(J_n) \to \infty \text{ and} \qquad (3.2)$$

$$\frac{J_n}{\mathrm{E}(J_n)} \to 1 \text{ in probability} \qquad (3.3)$$

where $J_n = \sum_{k=N}^{n} I_k$ for sufficiently large N, $I_k$being the indicator function of E_k.
In order to establish (3.2) consider

$$P(E_k) - P(E_k^*) = \mathrm{P}(F_k)\mathrm{P}(G_k) - P(F_k^*)\,P(G_k^*) \qquad (3.4)$$
$$= \mathrm{P}(F_k)\{\mathrm{P}(G_k) - P(G_k^*)\} + P(G_k^*)\{\mathrm{P}(F_k) - P(F_k^*)\}.$$

Therefore

$$|P(E_k) - P(E_k^*)| \leq \mathrm{P}(F_k)|\mathrm{P}(G_k) - P(G_k^*)| + P(G_k^*)|\mathrm{P}(F_k) - P(F_k^*)| = A_k \text{ say.} \qquad (3.5)$$

Observe

From the tail behavior of $\emptyset(x)$(d.f. of standard normal r.v.) that
(a) $P(E_k) \leq m_k \mathrm{P}\left(X_1 > d_{n_k^*}(x_1)\right) \sim$ (const.) $(\log k)^{-1/2}(\log n_k^*)^{-(x_1+1/2)}$as k→ ∞.
(b) $P(F_k^*) = 1 - \emptyset^{m_k}\left(d_{n_k^*}(x_1)\right) = 1 - \exp\left\{m_k \log\left\{1-\left(1-\emptyset\left(d_{n_k^*}(x_1)\right)\right)\right\}\right\}$
$= 1 - \exp\left\{-m_k\left(1-\emptyset\left(d_{n_k^*}(x_1)\right)\right)(1+o(1))\right\} \sim$ (const.)$(\log k)^{-1/2}(\log n_k^*)^{-(x_1+1/2)}$as k→ ∞, whenever $x_1 > -\frac{1}{2}$.
Similarly $P(G_k^*) \sim$ (const.) $(\log k)^{-1/2}(\log n_k^*)^{-(x_2+1/2)}$as k→ ∞, whenever $x_2 > -\frac{1}{2}$.
(2) By lemma 3.1 of Berman (1964)
$|\mathrm{P}(F_k) - P(F_k^*)| = \left|\mathrm{P}(F_k^c) - P\left({F_k^*}^c\right)\right| \leq (2\pi)^{-1}\sum_{j=1}^{m_k-1}|r(j)|\,(m_k - j)\left(1-r^2(j)\right)^{-1/2} \times \exp\left\{-{d_k^*}^2(x_1)/(1+|r(j)|)\right\}$
and similarly $|\mathrm{P}(G_k) - P(G_k^*)| \leq (2\pi)^{-1}\sum_{j=1}^{m_k-1}|r(j)|\,(m_k - j)\left(1-r^2(j)\right)^{-1/2} \times \exp\left\{-{d_k^*}^2(x_2)/(1+|r(j)|)\right\}$
By lemma 2.3 and lemma 2.4 we get $\sum_{k=N}^{\infty} A_k < \infty$ whenever either $(\log n)^{1+\gamma}$r(n)=O(1) as n→ ∞ for some γ >0 or $\sum_{j=1}^{\infty} r^2(j) < \infty$.
Further $P(F_k^*)\,P(G_k^*) \sim$ (const.)$(\log k)^{-1}\,(\log n_k^*)^{-(x_1+x_2+1)}$ as k→ ∞.
Since lim sup $\frac{n_k}{n_{k+1}}$<1, we have log $n_k^* >$ (const.) $k^{\frac{1}{1+x_1+x_2}}$ for large k. Hence
$P(F_k^*)\,P(G_k^*) \sim$ (const.)$(\log k)^{-1}\,k^{-1}$ as → ∞. Thus $\sum P(F_k^*)\,P(G_k^*) < \infty$. Hence from (3.5), (3.2) follows.
By Chebycheff's inequality we have

$$\mathrm{P}\left(\left|\frac{J_n}{EJ_n} - 1\right| > \varepsilon\right) \leq \frac{V(J_n)}{(\varepsilon^2(EJ_n)^2)} = \frac{\sum_{k=N}^{\infty} V(I_k) + 2\sum\sum_{N\leq k<l\leq n} Cov(I_k, I_l)}{\varepsilon^2(EJ_n)^2} \qquad (3.6)$$

Clearly $\sum_{k=N}^{\infty} V(I_k) \leq \sum_{k=N}^{\infty} E(I_k) = \mathrm{o}(EJ_n)^2$ as n→ ∞. Hence (3.3) will be established if we show

$\sup_n |\sum\sum_{N \le k < l \le n} Cov(I_k, I_l)| < \infty$ where N is a large positive integer. But this follows from lemma 2.5 and lemma 2.6. Hence the proof of the lemma is complete.

Lemma 3.3: For all $x_1, x_2 > -\frac{1}{2}$ with $x_1 + x_2 \ge \varepsilon^* - \frac{1}{2}$ and for all $\varepsilon > 0$ we have

$P(U_{n_k} > x_1 + \varepsilon, V_{n_k} > x_2 + \varepsilon$ i.o $) = 0$.

Proof: Since U_{n_k} and V_{n_k} are independent, we have

$P(U_{n_k} > x_1 + \varepsilon, V_{n_k} > x_2 + \varepsilon) = P(U_{n_k} > x_1 + \varepsilon)\, P(V_{n_k} > x_2 + \varepsilon)$

$< n_k^2\, P(X_1 > d_{n_k}(x_1 + \varepsilon))\, P(Y_1 > d_{n_k}(x_2 + \varepsilon))$

$<$ (const.) $n_k^2 \left(d_{n_k}(x_1 + \varepsilon)\right)^{-1} \left(d_{n_k}(x_2 + \varepsilon)\right)^{-1} \exp\left\{-\frac{1}{2} d_{n_k}^2(x_1 + \varepsilon) - -\frac{1}{2} d_{n_k}^2(x_2 + \varepsilon)\right\}$, $k \ge k_{16}$ by the tail behavior of the standard normal distribution.

$<$ (const.)$(\log n_k)^{(1 + x_1 + x_2 + 2\epsilon)}$. Hence $\sum P(U_{n_k} > x_1 + \varepsilon, V_{n_k} > x_2 + \varepsilon) < \infty$

Since $\frac{1}{2} + x_1 + x_2 + 2\epsilon > \epsilon^*$ as $x_1 + x_2 \ge \epsilon^* - \frac{1}{2}$. An application of Borel-Cantelli lemma completes the proof.

Proof of theorem 3.1: From lemmas 2.1 and2.2 it is clear that the limit set of (U_{n_k}, V_{n_k}) is contained in the square $\left\{(x_1, x_2) : -\frac{1}{2} \le x_1, x_2 \le \epsilon^*\right\}$. It follows from lemma 3.3 that the limit set is contained in S_2.

We conclude from lemmas 3.1 and 3.2 that every point of S_2 except the point $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ is a limit point. That the point $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ is also a limit point follows from continuity considerations.

ACKNOWLEDGEMENT

The first author is thankful to the Department of Science and Technology, New Delhi for the financial assistance.

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Source of Support: None Declared
Conflict of Interest: None Declared