On πgβ-Closed sets in Topological Spaces

A.Devika[1], N. Sakthilakshmi[2*] and R.Vani[3]

[1]Associate Professor, Dept. of Mathematics, P.S.G College of Arts and Science, Coimbatore (TN) INDIA

[2]Asst.Professor, Dept. of Maths with CA, Hindusthan College of Arts and Science, Coimbatore (TN) INDIA.

[3]Asst. Professor, Dept. of Mathematics with CA, P.S.G College of Arts and Science, Coimbatore (TN) INDIA

[*]Corresponding Address:

sakthinarayanasami@gmail.com

Research Article

Abstract: In this paper a new class of sets called πgβ_closed sets is introduced and its properties are studied. Further the notion of πgβ-$T_{½}$ space and πgβ-continuity are introduced.

Mathematics Subject Classification: 54A05, 54A40

Keywords: πgβ-closed, πgβ-open, πgβ-continuous, πgβ-$T_{½}$ spaces.

1. Introduction

Andrijevic [3] introduced a new class of generalized open sets in a topological space, the so-called β-open sets. This type of sets was discussed by Ekici and Caldas [11] under the name of γ-open sets. The class of β-open sets is contained in the class of semi-pre-open sets and contains all semi-open sets and preopen sets. The class of β-open sets generates the same topology as the class of preopen sets. Since the advent of these notations, several research paper with interesting results in different respects came to existence ([1,3,6,11,12,21,22,23]).Levine[16] introduced the concept of generalized closed sets in topological space and a class of topological spaces called $T_{½}$spaces. Extensive research on generalizing closedness was done in recent years as the notions of a generalized closed, generalized semi-closed α-generalized closed, generalized semi -pre-open closed sets were investigated in [2,7,16,18,19].The finite union of regular open sets is said to be π-open. The complement of a π-open set is said to be π-closed.

The aim of this paper is to study the notion of πgβ-closed sets and its various characterizations are given in this paper. In Section 3, the basic properties of πgβ-closed sets are studied. In section 4, the characterize πgβ-open sets is given. Finally in section 5, πgβ-continuous and πgβ-irresolute functions are discussed.

2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ) cl (A) and int (A) denote the closure of A and the interior of A respectively. (X, τ) will be replaced by X if there is no chance of confusion.

Let us recall the following definitions which shall be required later.

Definition 2.1: A subset A of a space (X, τ) is called

(1) a pre-open set [17] if A⊂int (cl (A)) and a preclosed set if cl(int(A))⊂A;

(2) a semi-open set [15] if A⊂cl (int(A)) and a semi-closed set if int(cl(A))⊂A

(3) a α-open set [20] if A⊂int(cl(int(A))) and a α-closed set if cl(int(cl(A)))⊂A

(4) a semi-preopen set[1] A⊂cl(intcl(A)) and a semi-pre-closed set if int(cl(int(A)))⊂A

(5) a regular open set if A=int(cl(A)) and a regular closed set if A=cl(int(A));

(6) b-open [3] or sp-open[8],γ-open[11]if A⊂cl(int(A))∪int(cl(A)).

The complement of a b-open set is said to be b-closed [3].The intersection of all b-closed sets of X containing A is called the b-closure of A and is denoted by bCl(A).The union of all b-open sets of X contained in A is called b-interior of A and is denoted by bInt(A).The family of all b-open (resp α-open, semi-open,preopen,β-open, b-closed,preclosed)subsets of a space X is denoted by bO(X)(resp αO(X),SO(X),PO(X),βO(X),bC(X),PC(X)) and the collection of all b-open subsets of X containing a fixed point x is denoted by bO(X,x).

The sets SO(X, x), α(X, x), PO(X, x), βO(X, x) are defined analogously.

Lemma 2.2[3]: Let A be a subset of a space X.Then

(1) bCl(A)=sCl(A)∩pCl(A)=A∪[Int(Cl(A))∩Cl(Int(A))];

(2) bInt(A)=sInt(A)∪pInt(A)=A∩[Int(Cl(A))∪Cl(Int(A))];

Definition 2.3: A subset A of a space (X, τ) is called

(1) a generalized closed (briefly g-closed) [16] if cl(A)⊂U whenever A⊂U and U is open.
(2) a generalized b-closed (briefly gb-closed) [13] if bcl(A)⊂U whenever A⊂U and U is open.
(3) πg-closed [10] if cl (A) ⊂U whenever A⊂U and U is π-open.
(4) πgp-closed [24] if pcl (A) ⊂U whenever A⊂U and U is π-open.
(5) πgα-closed [14] if αcl (A) ⊂U whenever A⊂U and U is π-open.
(6) πgsp-closed [25] if spcl (A) ⊂U whenever A⊂U and U is π-open.

Definition 2.4: A function f: (X, τ) ⟶ (Y, σ) is called
(1) π-irresolute [4] if f^{-1} (V) is π -closed in (X, τ) for every π -closed of (Y, σ);
(2) b- irresolute: [11] if for each b-open set V in Y, f^{-1} (V) is b-open in X:
(3) b-continuous: [11] if for each open set V in Y, f^{-1} (V) is b-open in X.

3. πgβ-closed sets

Definition 3.1: A is a subset of (X, τ) is called πgβ-closed if βcl (A) ⊂U whenever A⊂U and U is π-open in (X, τ). BY πGBC (τ) we mean the family of all πgβ-closed subsets of the space (X, τ).

Theorem 3.2:

1. Every closed set is πgβ-closed
2. Every g-closed is πgβ-closed
3. Every α-closed set is πgβ-closed
4. Every pre-closed set is πgβ-closed.
5. Every gβ-closed set is πgβ-closed.
6. Every πg-closed set is πgβ-closed.
7. Every πgp-closed set is πgβ-closed.
8. Every πgα–closed set is πgβ-closed.
9. Every πgs-closed set is πgβ-closed.
10. Every πgβ-closed set is πgsp-closed

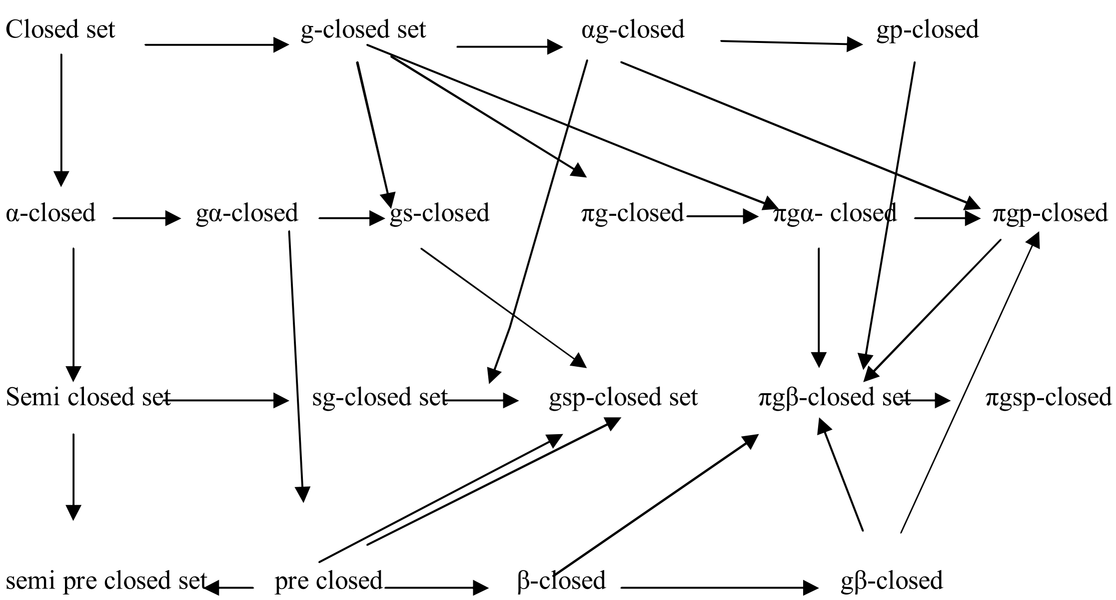
Proof: Straight forward converse of the above need not be true as seen in the following examples.

Example3.3: Consider X={a,b,c,d},τ = {ϕ,{a},{d},{a,d},{c,d},{a,c,d},X}.Let A= {c}.Then A is πgβ-closed but not closed, g-closed, α-closed,pre-closed,gβ-closed, πg-closed.

Example 3.4: Let X ={a,b,c,d,e} and τ ={ϕ ,{a,b},{c,d},{a,b,c,d},X}.Let A ={a}.Therefore A is πgβ-closed but not πgα-closed.

Example3.5: Let X={a,b,c,d,e} and τ = {ϕ ,{a,b},{c,d},{a,b,c,d},X}.
Let A = {a, b}.Therefore A is πgβ-closed but not πgp-closed.

Rewmark 3.6: The above discussions are summarized in the following diagram.



Theorem3.7: If A is π-open and πgβ-closed, then A is β-closed.

Proof: Let A is π-open and πgβ-closed.Let A⊂A where A is π-open. Since A is πgβ-closed, βcl (A) ⊂A.Then
A=βcl (A).Hence A is β-closed.

Theorem 3.8: Let A be a πgβ-closed in (X, τ).Then βcl (A)-A does not contain any non empty π-closed set.

Proof: Let F be a non empty π-closed set such that F⊂βcl(A)-A.since A is πgβ closed,A⊂X-F where X-F is π-open implies βcl(A)⊂X-F.Hence F⊂X-βcl(A).Now F⊂βcl(A)∩(X-βcl(A)) implies F=ϕ which is a contradiction. Therefore βcl (A) does not contain any non empty π–closed set.

Corollary3.9: Let A be πgβ-closed in (X, τ).Then A is β-closed iff βcl (A)-A is π- closed.

Proof: Let A be β-closed. Then βcl (A) =A. This implies βcl (A)-A=ϕ which is π-closed. Assume βcl (A)-A is π-closed. Then βcl (A)-A=ϕ. Hence βcl (A) =A.

Remark 3.10: Finite union of πgβ-closed sets need not be πgβ-closed.

Example 3.11: Consider X={a,b,c}, τ ={ ϕ ,{a},{b},{a,b},X}.Let A = {a},B={b}.Here A and B are πgβ-closed but A∪B ={a,b} is not πgβ-closed.

Remark 3.12: Finite intersection of πgβ-closed sets need not be πgβ-closed.

Example3.13: Consider X ={a,b,c,d},τ ={ ϕ,{a},{b},{a,b},{a,b.c},{a,b,d},X}.Let A= {a,b,c},B={a,b,d}.Here A and B are πgβ-closed but A∩B ={a,b} is not πgβ-closed.

Definition3.14 [5]: Let (X, τ) be a topological space A⊂X and x∊X is said to be β-limit point of A iff every β-open set containing x contains a point of A different from x.

Definition 3.15[5]: Let (X,τ) be a topological space,A⊂X.The set of all β-limit points of A is said to be β-derived set of A and is denoted by D_b [A]

Lemma 3.16 [5]: If D (A) = D_b [A], then we have cl (A) = βcl (A)

Lemma3.17 [5]: If D (A) ⊂D_b [A] for every subset A of X.Then for any subsets F and B of X, we have
Bcl (F∪B) = βcl(F)∪βcl(B)

Theorem3.18: Let A and B be πgβ-closed sets in (X, τ) such that D [A] ⊂D_b [A] and D [B] ⊂D_b [B]
Then A∪B is πgβ-closed.

Proof: Let U be π–open set such that A∪B⊂U.Since A and B are πgβ-closed sets we have β cl(A)⊂U and βcl(B)⊂U.Since D[A]⊂D_b [A] and D[B]⊂D_b [B] ,by lemma 3.16,cl(A)=βcl(A) and cl(B)=βcl(B).Thus
βcl(A∪B)⊂cl
(A∪B)=cl(A)∪cl(B)=βcl(A)∪βcl(B)⊂U.This implies A∪B is πgβ-closed.

Theorem3.19: If A is πgβ-closed set and B is any set such that A⊂B⊂βcl (A), then B is πgβ-closed set.

Proof: Let B⊂U and U be π-open. Given A⊂B.Then A⊂USince A is πgβ-closed, A⊂U implies βcl (A) ⊂U.By assumption it follows that βcl(B)⊂βcl(A)⊂U.Hence B is a πgβ-closed set.

4. πgβ-open sets

Definition 4.1: A set A⊂X is called πgβ-open if and only if its complement is πgβ-closed.

Remark4.2: βcl(X-A) =X-βint (A)
By πGBO (τ) we mean the family of all πgβ-open subsets of the space (X, τ).

Theorem4.3: If A⊂X is πgβ-open iff F⊂βint (A) whenever F is π-closed and F⊂A

Proof: Necessity: Let A be πgβ-open.Let F be π-closed and F⊂A.Then X-A⊂X-F where X-F is π-open. By assumption, βcl(X-A) ⊂X-F.By remark 4.2, X- β int (A) ⊂X-F.Thus F⊂βint (A).

Sufficiency: Suppose F is π-closed and F⊂A such that F⊂βint (A).Let X-A⊂U where U is π -open. Then X-U⊂A when X-U is π-closed. By hypothesis, X-U⊂β int (A).X-β int (A) ⊂U. βcl(X-A) ⊂U.Thus X-A is πgβ-closed and A is πgβ-open.

Theorem4.4: If βint (A) ⊂B⊂A and A is πgβ-open then B is πgβ-open.

Proof: Let βint (A) ⊂B⊂A.Thus X-A⊂X-B⊂βcl(X-A).Since X-A is πgβ-closed, by theorem 3.19, (X-A) ⊂(X-B) ⊂βcl(X-A) implies (X-B) is πgβ-closed.

Remark 4.5: For any A⊂X, βint (β cl (A)-A) = ϕ

Theorem 4.6: If A⊂X is πgβ-closed, then βcl (A)-A is πgβ-open.

Proof: Let A be πgβ-closed let F be π–closed set.F⊂βcl (A)-A. By theorem3.8, F=ϕ .By remark4.5,
βint (β (clA)-A) =ϕ .Thus F⊂βint (βcl (A)-A).Thus βcl (A)-A is πgβ-open.

Lemma4.7 [24]: Let A⊂X.If A is open or dense, then πO (A, τ /A) =V ∩ A such that V ∊ π O(X, τ).

Theorem 4.8: Let B ⊂ A ⊂ X where A is πgβ-closed and π-open set. Then B is πgβ-closed relative to A iff B is
Πgβ-closed in X.

Proof: Let B⊂A ⊂ X.Where A is πgβ-closed and π-open set. Let B be πgβ-closed in A.Let B⊂U where U is π-open in X.Since B⊂A, B=B∩A⊂U∩A, this implies βcl(B) =$βcl_A$ (B)⊂U∩A⊂U.Hence, B is πgβ -closed in X.

Let B be πgβ-closed in X.Let B⊂O where O is π-open in A.Then O=U∩A where U is π-open in X.This implies B⊂ O=U ∩ A ⊂ U.Since B is π g β –closed in X, βcl (B) ⊂U.Thus $βcl_A$ (B) =A∩β cl(B)⊂U A =O.Hence,B is πgβ-closed relative to A.

Corollary 4.9: Let A be π-open, πgβ-closed set. Then A∩F is πgβ-closed whenever F∊βC(X).

Proof: Since A is πgβ-closed and π-open, then βcl (A) ⊂A and thus A is β-closed. Hence A∩F is β-closed in X which implies A∩F is πgβ-closed in X.

Definition 4.10: A space (X,τ) is called a πgβ-$T_{1/2}$ space if every πgβ-closed set is β-closed.

Theorem 4.11:

(i)BO (τ) $\subset\pi$GBO (τ)
(ii) A space (X, τ) is $\pi g\beta$-$T_{1/2}$ iff BO (τ) =πGBO (τ).

Proof :(i) Let A be β-open, then X-A is β-closed so X-A is $\pi g\beta$-closed.Thus A is $\pi g\beta$-open.Hence
BO (τ) $\subset\pi$GBO (τ)

(ii)Necessity: Let (X, τ) be $\pi g\beta$-$T_{1/2}$ space. Let A$\epsilon\pi$GBO (τ).Then X-A is $\pi g\beta$-closed.By hypothesis-A is β-closed thus AϵBO (τ).Thus πGBO (τ) =BO (τ).

Suffiency: Let BO (T) =πGBO (τ).Let A be $\pi g\beta$-closed.Then X-A is $\pi g\beta$-open X-A$\epsilon\pi$GBO (τ).X-AϵBO (τ).Hence A is β-closed. This implies (X, τ) is $\pi g\beta$-$T_{1/2}$ space.

Lemma 4.12: Let A be a subset of (X, τ) and xϵX.Then x$\epsilon\beta$cl (A) iff V$\cap\phi$ for every β-open set V containing x.

Theorem 4.13: For a topological space (X, τ) the following are equivalent
(i)X is $\pi g\beta$-$T_{1/2}$ space.
(ii) Every singleton set is either π-closed or β-open.

Proof: To prove (i) $\Rightarrow$(ii): Let X be a $\pi g\beta$-$T_{1/2}$ space. Let xϵX and assuming that {x} is not π-closed. Then clearly X-{x} is not π-open. Hence X-{x} is trivially a $\pi g\beta$-closed.Since X is $\pi g\beta$-$T_{1/2}$ space-{x} is β-closed. Therefore {x} is β –open.

(ii) $\Rightarrow$ (i): Assume every singleton of X is either π-closed or β-open. Let A be a $\pi g\beta$-closed set. Let {x} ϵ β cl (A).

Case (i): Let {x} be π closed. Suppose {x} does not belong to A. Then {x} ϵ β cl (A)-A.By theorem3.8, {x} ϵ A. Hence β cl (A) $\subset$ A.

Case (ii): Let {x} be β-open. Since {x} ϵ β cl (A), we have {x} $\cap$ A $\neq\phi$ $\Rightarrow$ {x} ϵ A. Therefore β cl (A) $\subset$ A. Therefore A is β-closed.

5. $\pi g\beta$-continuous and $\pi g\beta$-irresolute functions

Definition 5.1:A function f:(X,τ)$\rightarrow$(Y,σ) is called $\pi g\beta$-continuous if every f^{-1} (V) is $\pi g\beta$-closed in (X,τ) for every closed set V of (Y,σ).

Definition 5.2:A function f(X,τ)$\longrightarrow$(y,σ)is called $\pi g\beta$-irresolute if f^{-1} (V) is $\pi g\beta$-closed in (X,τ) for every $\pi g\beta$- closed set V in (Y,σ)

Proposition 5.3: Every $\pi g\beta$-irresolute function is $\pi g\beta$-continuous.

Remark 5.4: Converse of the above need not be true.

Proof: Let f :(X, τ) $\longrightarrow$ (Y, σ) be a $\pi g\beta$-continuous function and V be a $\pi g\beta$-closed in (Y, σ).

But every $\pi g\beta$-closed set need not be closed in (Y, σ).So there exists some sets which is not closed in (Y, σ).By definition there exits some sets which are not $\pi g\beta$-closed in (X, τ), which implies f is not $\pi g\beta$-irresolute.

Remark 5.6: Composition of two $\pi g\beta$-continuous functions need not be $\pi g\beta$-continuous.

Proof: Let f :(X τ) $\longrightarrow$ (X, σ) and g :(X, σ) $\longrightarrow$ (X, η) be two $\pi g\beta$-continuous functions.

Let V be a closed set in (X, η).Since g is a $\pi g\beta$-continuous function, g^{-1}(V) is $\pi g\beta$-closed set in (X, σ).But every $\pi g\beta$-closed is not closed. Therefore there exist some sets in (X, σ) which is not $\pi g\beta$-closed in (X, τ).

Hence g_0f is not $\pi g\beta$-continuous.

Definition5.8: A function f: X $\longrightarrow$ Y is said to be pre β-closed if f (U) is β-closed in Y for each β-closed set in X.

Proposition 5.9: Let f :(X,τ)$\rightarrow$(Y,σ) be π-irresolute and pre β-closed map. Then f (A) is $\pi g\beta$-closed in Y for every $\pi g\beta$-closed set A of X.

Proof: Let A be $\pi g\beta$–closed in X.Let f(A)$\subset$V where V is π-open in Y.Then A$\subset$ f^{-1} (V) and A is $\pi g\beta$–closed in X implies βcl(A)$\subset f^{-1}$ (V).Hence f(β cl(A))$\subset$V.Since f is pre β-closed, βcl(A)$\subset\beta$cl(f(βcl(A)))=f(βcl(A))$\subset$V.Hence f(A) is $\pi g\beta$-closed in Y.

Definition 5.10: A topological space X is a $\pi g\beta$-space if every $\pi g\beta$-closed set is closed.

Proposition 5.11: Every $\pi g\beta$ space is $\pi g\beta$-$T_{1/2}$ space.

Theorem 5.12: Let f :(X,τ) $\longrightarrow$ (Y,σ) be a function.
(1)If is $\pi g\beta$-irresolute and X is $\pi g\beta$- $T_{1/2}$ space, then f is β-irresolute.
(2)If f is $\pi g\beta$-continuous and X is $\pi g\beta$-$T_{1/2}$ space, then f is β-continuous.

Proof :(1) Let V be β-closed in Y.Since f is $\pi g\beta$-irresolute, f^{-1} (V) is $\pi g\beta$-closed in X.Since X is $\pi g\beta$-$T_{1/2}$ space, f^{-1}(V) is β-closed in X.Hence f is β-irresolute.

(2)Let V be closed in Y.Since f is $\pi g\beta$ continuous, f^{-1}(V) is $\pi g\beta$-closed in X.By assumption, it is β-closed. Therefore f is β-continuous.

Definition5.13 [14]: A function f: (X,τ) $\longrightarrow$ (Y,σ) is π-open map in Y for every π-open in X

Theorem5.14: If the bijective f: (X,τ) $\longrightarrow$ (Y,σ) is β-irresolute and π-open map, then f is $\pi g\beta$-irresolute.

Proof: Let V be $\pi g\beta$-closed in Y.Let f^{-1}(V)$\subset$U where U is π-open in X.Then V $\subset$ f(U)and f(U)is π-open implies βcl(V)$\subset$f(U).Since f is β-irresolute,(f^{-1}(βcl(V))) is β-closed. Hence βcl(f^{-1}(V))$\subset\beta$cl(f^{-1}(βcl(V)))=f^{-1}(βcl(V))$\subset$U. Therefore is $\pi g\beta$-irresolute.

Theorem 5.15: If f: X$\longrightarrow$Y is π-open, β-irresolute, pre β-closed subjective function. If X is $\pi g\beta$-$T_{1/2}$ space, then Y is $\pi g\beta$-$T_{1/2}$ space.

Proof: Let V be πgβ-closed in Y.Let $f^{-1}(V) \subset U$ where U is π-open in X.Then $F \subset f(U)$ and F is a πgβ–closed set in Y implies β cl(F)⊂f(U).Since f is β-irresolute, β cl(f^{-1} (F))⊂β cl(f^{-1}β cl(F)))= f^{-1}(β cl(F)) ⊂ U.Therefore $f^{-1}(F) = F$ is β-closed in Y.Hence Y is πgβ-$T_{1/2}$ space.

References

[1] D.Andrijevic, Semipreopen sets, Mat.Vesnik., 38;24-32, 1986.

[2] S.P.Arya and T.M.Nour, Characterizations of s-normal spaces, Indian J.Pure Appl.Math., 21; no.8, 717-719, 1990.

[3] D.Andrijevic, On β-open sets, Mat.Vesnik 48; 59-64, 1996.

[4] G.Aslim, A.Caksu Guler and T.Noiri, on πgs-closed sets in topological Spaces, Acta Math.Hungar., 112(4); 275-283, 2006.

[5] Ahmad-Al-Omari and Mohd.Salmi Md Noorani, On Generalized β –closed Sets, Bull.Malays.Math.Sci.Soc., 32(1); 19-30, 2009.

[6] M.Caldas and S.Jafari.On some applications of β-open set in topological spaces, Kochi J.Math., 2;11-19, 2007.

[7] J.Dontchev, On generalizing semi-preopen sets, Mem.Fac.Kochi Univ.Ser.A Math., 16; 35-48, 1995.

[8] J.Dontchev and M.Przemski, On the various decompositions of continuous and some weakly continuous functions, Acta Math.Hungar.71; 109-120, 1996.

[9] J.Dontchevand M.Ganster, Onδ-generalized closed sets and $T_{3/4}$-spaces, Mem.Fac.Kochi Univ.Ser.A Math., 17; 15-31, 1996.

[10] J.Dontchev and T.Noiri,Quasi Normal Spaces and π g-closed sets,Acta Math.Hungar.,89(3); 211-219, 2000.

[11] E.Ekici and M.Caldas, Slightly-continuous functions, Bol.Soc.Parana.Mat. (3) 22; 63-74, 2004.

[12] M.Ganster and M.Steiner, On some questions about β –open sets, Questions Answers Gen.Topology 25; 45-52, 2007.

[13] M. Ganster and M.Steiner, On b τ-closed sets, Appl. Gen.Topology 8; 243-247, 2007.

[14] C. Janaki, Studies on πgα-closed sets in Topology, Ph.D Thesis, Bharathiar University, Coimbatore, 2009.

[15] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer.Math.Monthly70; 36-41, 1963.

[16] N. Levine, Generalized closed sets in Topology, Rend.Circ.Mat.Palermo (2)19; 89-96, 1970.

[17] A.S Mashhour, M.E.Abd El-Monsef and S.N.El-Deep, On pre continuous and weak pre-continuous mappings, Proc.Math.Phys.Soc.Egypt No.53; 47-53, 1982.

[18] H.Maki, R.Devi and K.Balachandran, Associated topologies of generalized α-closed sets and α-generalized closed sets, Mem.Fac.Sci.Kochi Univ.Ser.A Math.15; 51-63, 1994.

[19] H.Maki, J.Umehara and T.Noiri, every topological space is pre-T $T_{1/2}$, Mem.Fac.Sci.Kochi Univ. Ser. A. Math., 17; 33-42, 1996.

[20] O. Njastad, On some classes of nearly open sets, Pacific J.Math.15; 961-970, 1965.

[21] A. A. Nasef, On b-locally closed sets and related topics, Chaos Solitons Fractals 12; 1909-1915, 2001.

[22] A .A. Nasef, On Some properties of contra-continuous functions, Chaos Solitons Fractals 24; 471-477, 2005.

[23] J. H. Park, Strongly θ-b-continuous functions, Acta Math.Hungar.110; 347-359, 2006.

[24] J. H. Park, On πgp-closed sets in Topological Spaces, Indian.J.Pure Appl.Math., (To appear).

[25] M. S. Sarsak and N. Rajesh, π-Generalized semi-preclosed Sets, International Mathematical Forum, 5;573-578, 2010.