On Certain Conditions of Geometric Functions for the Generalized Hypergeometric Functions

Chena Ram[1] and Garima Agarwal[2]

[2]Research Scholar, Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur, Rajasthan, INDIA.

Corresponding Addresses:

[1]crc2007@rediffmail.com, [2]garima.msc.phd@gmail.com

Research Article

Abstract: The aim of the present paper is to obtain sufficient conditions for the function $z\ {}_3\mathrm{R}_2(\varphi, a, b; c, d; k; z)$ for its belongingness to certain subclasses of starlike and convex functions. Similar results, using integral operator, are also obtained.

Key words: Starlike Functions, Convex Functions, Hypergeometric Functions, Clausenian Hypergeometric functions, Integral operator.

1. Introduction

Let T denote the class consisting of functions $f(z)$, given by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0), \tag{1.1}$$

which are analytic and univalent in the open unit disk U = $\{z: |z| \leq 1\}$.

Let $T(\lambda, \alpha)$ is the subclass of the class T. Which has functions satisfying the condition

$$Re\left\{\frac{zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)}\right\} > \alpha, \tag{1.2}$$

for some $\alpha(0 \leq \alpha < 1)$, $\lambda(0 \leq \lambda < 1)$ and for all $z \in U$.

Also let $C(\lambda, \alpha)$ denote the subclass of T constituted by functions those satisfy the condition

$$Re\left\{\frac{f'(z)+zf''(z)}{f'(z)+\lambda zf''(z)}\right\} > \alpha, \tag{1.3}$$

for some $\alpha(0 \leq \alpha < 1)$, $\lambda(0 \leq \lambda < 1)$ and for all $z \in U$.

From (1.2) and (1.3), we have

$$f(z) \in C(\lambda, \alpha) \Leftrightarrow zf'(z) \in T(\lambda, \alpha). \tag{1.4}$$

We note that $T(0, \alpha) = T^*(\alpha)$, the class of starlike functions of order $\alpha(0 \leq \alpha < 1)$ and $C(0, \alpha) = C(\alpha)$, the class of convex functions of order $\alpha(0 \leq \alpha < 1)$ (see Silverman[5]).

Following definitions [3] will be required in proving the main results:

$${}_3R_2^k(z) = {}_3\mathrm{R}_2(\varphi, a, b; c, d; k; z) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(\varphi)_n \Gamma(a+kn)\Gamma(b+kn)}{\Gamma(c+kn)\Gamma(d+kn)} \frac{z^n}{(n)!}, \tag{1.5}$$

where $|z| < 1, k > 0.$

If we set $b = d$, then the generalized hypergeometric function ${}_3R_2^k(z)$ reduces to the result due to Virchenko, Kalla and Zamel [6], as

$${}_2R_1^k(z) = {}_2\mathrm{R}_1(\varphi, a; c; k; z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{(\varphi)_n \Gamma(a+kn)}{\Gamma(c+kn)} \frac{z^n}{(n)!}. \tag{1.6}$$

For $k = 1$, (1.5) reduces to the Clausenian hypergeometric function studied by Saxena and Kalla [4]

$${}_3\mathrm{F}_2(\varphi, a, b; c, d; z) = \sum_{n=0}^{\infty} \frac{(\varphi)_n (a)_n (b)_n}{(c)_n (d)_n} \frac{z^n}{(n)!}, \tag{1.7}$$

and for $k = 1$, (1.6) reduces to the Gauss hypergeometric function

$${}_2\mathrm{F}_1(\varphi, a; c; z) = \sum_{n=0}^{\infty} \frac{(\varphi)_n (a)_n}{(c)_n} \frac{z^n}{(n)!}. \tag{1.8}$$

In the present paper we shall use the following lemmas, see Altintas and Owa [1].

Lemma 1. A function $f(z)$ of the form (1.1) is in the class $T(\lambda, \alpha)$, if and only if

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)a_n \leq (1-\alpha). \quad (1.9)$$

Lemma 2. A function $f(z)$ of the form (1.1) is in the class $C(\lambda,\alpha)$, if and only if

$$\sum_{n=2}^{\infty}n(n-\lambda\alpha n-\alpha+\lambda\alpha)a_n \leq (1-\alpha). \quad (1.10)$$

Main Results

Theorem 2.1. If $k \in R, (k>0)$, then $z\,{}_3\mathrm{R}_2(\varphi,a,b;c,d;k;z)$ is in the class $T(\lambda,\alpha)$, if and only if

$$(1-\lambda\alpha){}_3\mathrm{R}_2(\varphi+1,a+k,b+k;c+k,d+k;k;1)$$
$$+(1-\alpha){}_3\mathrm{R}_2(\varphi,a,b;c,d;k;1)\frac{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)}{(\varphi)\Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)} \leq 0. \quad (2.1)$$

Proof: Since

$z\,{}_3\mathrm{R}_2(\varphi,a,b;c,d;k;z)=$

$$z-\left|\frac{(\varphi)\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\right|\sum_{n=2}^{\infty}\frac{(\varphi+1)_{n-2}\Gamma(a+k+k(n-2))\Gamma(b+k+k(n-2))}{\Gamma(c+k+k(n-2))\Gamma(d+k+k(n-2))}\frac{z^n}{(1)_{n-1}}, \quad (2.2)$$

by Lemma 1, it is sufficient to prove that

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(\varphi+1)_{n-2}\Gamma(a+k+k(n-2))\Gamma(b+k+k(n-2))}{\Gamma(c+k+k(n-2))\Gamma(d+k+k(n-2))(1)_{n-1}} \leq \left|\frac{\Gamma(a)\Gamma(b)}{(\varphi)\Gamma(c)\Gamma(d)}\right|(1-\alpha).$$

Now we have

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(\varphi+1)_{n-2}\Gamma(a+k+k(n-2))\Gamma(b+k+k(n-2))}{\Gamma(c+k+k(n-2))\Gamma(d+k+k(n-2))(1)_{n-1}}$$

$$=(1-\lambda\alpha)\sum_{n=2}^{\infty}(n)\frac{(\varphi+1)_{n-2}\Gamma(a+k+k(n-2))\Gamma(b+k+k(n-2))}{\Gamma(c+k+k(n-2))\Gamma(d+k+k(n-2))(1)_{n-1}}$$

$$-(1-\lambda)\alpha\sum_{n=2}^{\infty}\frac{(\varphi+1)_{n-2}\Gamma(a+k+k(n-2))\Gamma(b+k+k(n-2))}{\Gamma(c+k+k(n-2))\Gamma(d+k+k(n-2))(1)_{n-1}}$$

i.e. $$=(1-\lambda\alpha)\sum_{n=0}^{\infty}(n+2)\frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n+1}}$$

$$-(1-\lambda)\alpha\sum_{n=0}^{\infty}\frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n+1}}$$

i.e $$=(1-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_n}$$

$$+(1-\alpha)\sum_{n=1}^{\infty}\frac{(\varphi)_n\Gamma(a+kn)\Gamma(b+kn)}{(\varphi)\Gamma(c+kn)\Gamma(d+kn)(1)_n},$$

which on using (1.5) yields

$$=(1-\lambda\alpha){}_3\mathrm{R}_2(\varphi+1,a+k,b+k;c+k,d+k;k;1)\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(d+k)}$$
$$+(1-\alpha)\left[{}_3\mathrm{R}_2(\varphi,a,b;c,d;k;1)\frac{\Gamma(a)\Gamma(b)}{(\varphi)\Gamma(c)\Gamma(d)}-\frac{\Gamma(a)\Gamma(b)}{(\varphi)\Gamma(c)\Gamma(d)}\right] \leq \left|\frac{\Gamma(a)\Gamma(b)}{(\varphi)\Gamma(c)\Gamma(d)}\right|(1-\alpha).$$

The theorem is completely proved.

Corollary 1. $z\,{}_3\mathrm{F}_2(\varphi,a,b;c,d;z)$ is in the class $T(\lambda,\alpha)$, if and only if

$$(1-\lambda\alpha){}_3\mathrm{F}_2(\varphi+1,a+1,b+1;c+1,d+1;1)$$
$$+(1-\alpha){}_3\mathrm{F}_2(\varphi,a,b;c,d;1)\frac{(c)(d)}{(\varphi)(a)(b)} \leq 0. \quad (2.3)$$

Proof. If we set $k=1$ in (2.1), the proof is completed.

Corollary 2. $z\,{}_2\mathrm{R}_1(\varphi,a;c;k;z)$ is in the class $T(\lambda,\alpha)$, if and only if

$$(1-\lambda\alpha){}_2\mathrm{R}_1(\varphi+1,a+k;c+k;k;1)$$
$$+(1-\alpha){}_2\mathrm{R}_1(\varphi,a;c;k;1)\frac{\Gamma(a)\Gamma(c+k)}{(\varphi)\Gamma(c)\Gamma(a+k)} \leq 0. \quad (2.3)$$

Proof. If we set $b = d$ in (2.1), the proof is completed.

Theorem 2.2. If $k \in R, (k > 0)$, then
$R_2(\varphi, a, b; c, d; k; z) = z[2 - {}_3R_2(\varphi, a, b; c, d; k; z)]$ is in the class $T(\lambda, \alpha)$, if and only if

$$(1-\lambda\alpha)\,{}_3R_2(\varphi+1, a+k, b+k; c+k, d+k; k; 1)\frac{(\varphi)\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(d+k)} + (1-\alpha)[{}_3R_2(\varphi, a, b; c, d; k; 1) - 1]\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)} \le \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)}(1-\alpha). \quad (2.5)$$

Proof.

$$R_2(\varphi, a, b; c, d; k; z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} \frac{(\varphi)_{n-1}\Gamma(a+k(n-1))\Gamma(b+k(n-1))}{\Gamma(c+k(n-1))\Gamma(d+k(n-1))(1)_{n-1}} z^n \quad (2.6)$$

By Lemma 1, we need only to show that

$$\sum_{n=2}^{\infty} (n - \lambda\alpha n - \alpha + \lambda\alpha) \frac{(\varphi)_{n-1}\Gamma(a+k(n-1))\Gamma(b+k(n-1))}{\Gamma(c+k(n-1))\Gamma(d+k(n-1))(1)_{n-1}} \le \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)}(1-\alpha).$$

Now,

$$\sum_{n=2}^{\infty} (n - \lambda\alpha n - \alpha + \lambda\alpha) \frac{(\varphi)_{n-1}\Gamma(a+k(n-1))\Gamma(b+k(n-1))}{\Gamma(c+k(n-1))\Gamma(d+k(n-1))(1)_{n-1}}$$

$$= \sum_{n=2}^{\infty} [(n-1)(1-\lambda\alpha) + (1-\alpha)] \frac{(\varphi)_{n-1}\Gamma(a+k(n-1))\Gamma(b+k(n-1))}{\Gamma(c+k(n-1))\Gamma(d+k(n-1))(1)_{n-1}}$$

i.e. $$= (1-\lambda\alpha)\varphi \sum_{n=1}^{\infty} \frac{(\varphi+1)_{n-1}\Gamma(a+k+k(n-1))\,\Gamma(b+k+k(n-1))}{\Gamma(c+k+k(n-1))\,\Gamma(d+k+k(n-1))\,(1)_{n-1}} + (1-\alpha)\sum_{n=1}^{\infty} \frac{(\varphi)_n\Gamma(a+kn)\Gamma(b+kn)}{\Gamma(c+kn)\Gamma(d+kn)(1)_n}$$

$$= (1-\lambda\alpha)\varphi \sum_{n=0}^{\infty} \frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_n} + (1-\alpha)\left[\sum_{n=0}^{\infty} \frac{(\varphi)_n\Gamma(a+kn)\Gamma(b+kn)}{\Gamma(c+kn)\Gamma(d+kn)(1)_n} - \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)}\right]$$

Hence,

$$(1-\lambda\alpha)\,{}_3R_2(\varphi+1, a+k, b+k; c+k, d+k; k; 1)\frac{(\varphi)\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(d+k)} + (1-\alpha)[{}_3R_2(\varphi, a, b; c, d; k; 1) - 1]\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)} \le \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)}(1-\alpha).$$

The proof is completed.

Corollary 3. $F_2(\varphi, a, b; c, d; z) = z[2 - {}_3F_2(\varphi, a, b; c, d; z)]$ is in the class $T(\lambda, \alpha)$ if and only if

$$(1-\lambda\alpha)\,{}_3F_2(\varphi+1, a+1, b+1; c+1, d+1; 1)\frac{(\varphi)(a)(b)}{(c)(d)} + (1-\alpha)[{}_3F_2(\varphi, a, b; c, d; 1) - 1] \le (1-\alpha). \quad (2.7)$$

Proof. If we set $k = 1$ in (2.5), the proof is completed.

Corollary 4. $R_1(\varphi, a; c; k; z) = z[2 - {}_2R_1(\varphi, a; c; k; z)]$ is in the class $T(\lambda, \alpha)$ if and only if

$$(1-\lambda\alpha)\,{}_2R_1(\varphi+1, a+k; c+k; k; 1)\frac{(\varphi)\Gamma(a+k)}{\Gamma(c+k)} + (1-\alpha)[{}_2R_1(\varphi, a; c; k; 1) - 1]\frac{\Gamma(a)}{\Gamma(c)} \le \frac{\Gamma(a)}{\Gamma(c)}(1-\alpha). \quad (2.8)$$

Proof. If we set $b = d$ in (2.5), the proof is completed.

Theorem 2.3. If $k \in R, (k > 0)$, then $z\,{}_3R_2(\varphi, a, b; c, d; k; z)$ is in the class $C(\lambda, \alpha)$, if and only if

$$(1-\lambda\alpha)\,{}_3R_2(\varphi+2, a+2k, b+2k; c+2k, d+2k; k; 1)\frac{(\varphi+1)\Gamma(a+2k)\Gamma(b+2k)}{\Gamma(c+2k)\Gamma(d+2k)} + (3-\alpha-2\lambda\alpha)\,{}_3R_2(\varphi+1, a+k, b+k; c+k, d+k; k; 1)\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(d+k)} + (1-\alpha)\,{}_3R_2(\varphi, a, b; c, d; k; 1)\frac{\Gamma(a)\Gamma(b)}{(\varphi)\Gamma(c)\Gamma(d)} \le 0. \quad (2.9)$$

Proof. From (2.2), we have

$z\,{}_3\mathrm{R}_2(\varphi, a, b; c, d; k; z) =$

$$z - \left|\frac{(\varphi)\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\right| \sum_{n=2}^{\infty} \frac{(\varphi+1)_{n-2}\Gamma(a+k+k(n-2))\Gamma(b+k+k(n-2))}{\Gamma(c+k+k(n-2))\Gamma(d+k+k(n-2))} \frac{z^n}{(1)_{n-1}}$$

By Lemma 2, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n(n-\lambda\alpha n-\alpha+\lambda\alpha) \frac{(\varphi+1)_{n-2}\Gamma(a+k+k(n-2))\Gamma(b+k+k(n-2))}{\Gamma(c+k+k(n-2))\Gamma(d+k+k(n-2))(1)_{n-1}} \leq \left|\frac{\Gamma(a)\Gamma(b)}{(\varphi)\Gamma(c)\Gamma(d)}\right|(1-\alpha)$$

Now we have

$$\sum_{n=0}^{\infty}(n+2)[(n+2)(1-\lambda\alpha)-\alpha(1-\lambda)] \frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n+1}}$$

$$= (1-\lambda\alpha) \sum_{n=0}^{\infty}(n+1)^2 \frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n+1}}$$

$$+(2-\alpha-\lambda\alpha) \sum_{n=0}^{\infty}(n+1) \frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n+1}}$$

$$+(1-\alpha) \sum_{n=0}^{\infty} \frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n+1}}$$

$$= (1-\lambda\alpha) \sum_{n=0}^{\infty}(n+1) \frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n}}$$

$$+(2-\alpha-\lambda\alpha) \sum_{n=0}^{\infty} \frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n}}$$

$$+(1-\alpha) \sum_{n=0}^{\infty} \frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n+1}}$$

$$= (1-\lambda\alpha) \sum_{n=0}^{\infty}(n) \frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n}}$$

$$+(3-\alpha-2\lambda\alpha) \sum_{n=0}^{\infty} \frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n}}$$

$$+(1-\alpha) \sum_{n=1}^{\infty} \frac{(\varphi)_n\Gamma(a+kn)\Gamma(b+kn)}{(\varphi)(c+kn)\Gamma(d+kn)(1)_{n}}$$

$$= (1-\lambda\alpha) \sum_{n=0}^{\infty} \frac{(\varphi+1)(\varphi+2)_n\Gamma(a+2k+kn)\Gamma(b+2k+kn)}{\Gamma(c+2k+kn)\Gamma(d+2k+kn)(1)_{n}}$$

$$+(3-\alpha-2\lambda\alpha) \sum_{n=0}^{\infty} \frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n}}$$

$$+(1-\alpha)\left[\sum_{n=0}^{\infty} \frac{(\varphi)_n\Gamma(a+kn)\Gamma(b+kn)}{(\varphi)\Gamma(c+kn)\Gamma(d+kn)(1)_{n}} - \frac{\Gamma(a)\Gamma(b)}{(\varphi)\Gamma(c)\Gamma(d)}\right]$$

$$=(1-\lambda\alpha)\,{}_3\mathrm{R}_2(\varphi+2, a+2k, b+2k; c+2k, d+2k; k; 1) \frac{(\varphi+1)\Gamma(a+2k)\Gamma(b+2k)}{\Gamma(c+2k)\Gamma(d+2k)}$$

$$+(3-\alpha-2\lambda\alpha)\,{}_3\mathrm{R}_2(\varphi+1, a+k, b+k; c+k, d+k; k; 1) \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(d+k)}$$

$$+(1-\alpha)\,{}_3\mathrm{R}_2(\varphi, a, b; c, d; k; 1) \frac{\Gamma(a)\Gamma(b)}{(\varphi)\Gamma(c)\Gamma(d)} - (1-\alpha) \frac{\Gamma(a)\Gamma(b)}{(\varphi)\Gamma(c)\Gamma(d)}$$

$$\leq \left|\frac{\Gamma(a)\Gamma(b)}{(\varphi)\Gamma(c)\Gamma(d)}\right|(1-\alpha).$$

The proof is completed.

Corollary 5. $z\,{}_3\mathrm{F}_2(\varphi, a, b; c, d; z)$ is in the class $C(\lambda, \alpha)$ if and only if

$$(1-\lambda\alpha)\,{}_3\mathrm{F}_2(\varphi+2, a+2, b+2; c+2, d+2; 1) \frac{(\varphi+1)(a)(a+1)(b)(b+1)}{(c)(c+1)(d)(d+1)}$$

$$+(3-\alpha-2\lambda\alpha){}_3\mathrm{F}_2(\varphi+1,a+1,b+1;c+1,d+1;1)\frac{(\mathrm{a})(\mathrm{b})}{(\mathrm{c})(\mathrm{d})}$$
$$+(1-\alpha){}_3\mathrm{F}_2(\varphi,a,b;c,d;1)\frac{1}{(\varphi)}\leq 0. \quad (2.10)$$

Proof. If we set $k=1$ in (2.9), the proof is completed.

Corollary (6). $z\,{}_2\mathrm{R}_1(\varphi,a;c;k;z)$ is in the class $C(\lambda,\alpha)$ if and only if

$$(1-\lambda\alpha){}_2\mathrm{R}_1(\varphi+2,a+2k;c+2k;k;1)\left[\frac{(\varphi+1)\Gamma(a+2k)}{\Gamma(c+2k)}\right]$$
$$+(3-\alpha-2\lambda\alpha){}_2\mathrm{R}_1(\varphi+1,a+k;c+k;k;1)\frac{\Gamma(\mathrm{a+k})}{\Gamma(\mathrm{c+k})}$$
$$+(1-\alpha){}_2\mathrm{R}_1(\varphi,a;c;k;1)\frac{\Gamma(\mathrm{a})}{(\varphi)\Gamma(c)}\leq 0. \ (2.11)$$

Proof. If we set $b=d$ in (2.9), the proof is completed.

Theorem 2.4. If $k\in R,(k>0)$, then
$\mathrm{R}_2(\varphi,a,b;c,d;k;z)=\ z[2\text{-}\,{}_3\mathrm{R}_2(\varphi,a,b;c,d;k;z)]$ is in the class $C(\lambda,\alpha)$ if and only if

$$(1-\lambda\alpha){}_3\mathrm{R}_2(\varphi+2,a+2k,b+2k;c+2k,d+2k;k;1)\frac{(\varphi+1)\Gamma(a+2k)\Gamma(b+2k)}{\Gamma(c+2k)\Gamma(d+2k)}$$
$$+(3-\alpha-2\lambda\alpha){}_3\mathrm{R}_2(\varphi+1,a+k,b+k;c+k,d+k;k;1)\frac{\Gamma(a+k)\Gamma(b+k)}{(\varphi)\Gamma(c+k)\Gamma(d+k)}$$
$$+(1-\alpha)\,{}_3\mathrm{R}_2(\varphi,a,b;c,d;k;1)\leq 0. \qquad (2.12)$$

Proof. By (2.6) we have

$$\mathrm{R}_2(\varphi,a,b;c,d;k;z)=z-\sum_{n=2}^{\infty}\frac{(\varphi)\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\frac{(\varphi+1)_{n-2}\Gamma\big(a+k+k(n-2)\big)\Gamma\big(b+k+k(n-2)\big)}{\Gamma\big(c+k+k(n-2)\big)\Gamma\big(d+k+k(n-2)\big)(1)_{n-1}}z^n \quad (2.13)$$

By Lemma 2, we need only to show that

$$\sum_{n=2}^{\infty}n(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(\varphi+1)_{n-2}\Gamma\big(a+k+k(n-2)\big)\Gamma\big(b+k+k(n-2)\big)}{\Gamma\big(c+k+k(n-2)\big)\Gamma\big(d+k+k(n-2)\big)(1)_{n-1}}\leq\left|\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)}\right|(1-\alpha).$$

$$\sum_{n=0}^{\infty}(n+2)[(n+2)(1-\lambda\alpha)-\alpha(1-\lambda)]\frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n+1}}$$

$$=(1-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(n+1)^2(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n+1}}$$
$$+(2-\alpha-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(n+1)(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n+1}}$$
$$+(1-\alpha)\sum_{n=0}^{\infty}\frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n+1}}$$

$$=(1-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(n+1)(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_n}$$
$$+(2-\alpha-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_n}$$
$$+(1-\alpha)\sum_{n=0}^{\infty}\frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_{n+1}}$$

$$=(1-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(n)(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_n}$$
$$+(3-\alpha-2\lambda\alpha)\sum_{n=0}^{\infty}\frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)}{\Gamma(c+k+kn)\Gamma(d+k+kn)(1)_n}$$
$$+(1-\alpha)\sum_{n=1}^{\infty}\frac{(\varphi+1)_{n-1}\Gamma(a+kn)\Gamma(b+kn)}{\Gamma(c+kn)\Gamma(d+kn)(1)_n}$$

$$(1-\lambda\alpha){}_3\mathrm{R}_2(\varphi+2,a+2k,b+2k;c+2k,d+2k;k;1)\frac{(\varphi+1)\Gamma(a+2k)\Gamma(b+2k)}{\Gamma(c+2k)\Gamma(d+2k)}$$

$$+(3-\alpha-2\lambda\alpha)_3\mathrm{R}_2(\varphi+1,a+k,b+k;c+k,d+k;k;1)\frac{\Gamma(a+k)\Gamma(b+k)}{(\varphi)\Gamma(c+k)\Gamma(d+k)}$$
$$+(1-\alpha)\left[{}_3\mathrm{R}_2(\varphi,a,b;c,d;k;1)-1\right]\frac{\Gamma(a)\Gamma(b)}{(\varphi)\Gamma(c)\Gamma(d)}\leq\left|\frac{\Gamma(a)\Gamma(b)}{(\varphi)\Gamma(c)\Gamma(d)}\right|(1-\alpha).$$

The Theorem is completely proved.

Corollary 7. $\mathrm{F}_2(\varphi,a,b;c,d;z)=z[2-{}_3\mathrm{F}_2(\varphi,a,b;c,d;z)]$ is in the class $C(\lambda,\alpha)$ if and only if

$$(1-\lambda\alpha)_3\mathrm{F}_2(\varphi+2,a+2,b+2;c+2,d+2;1)\frac{(\varphi+1)(a+1)(b+1)(\varphi)(a)(b)}{(c+1)(d+1)(c)(d)}$$
$$+(3-\alpha-2\lambda\alpha)_3\mathrm{F}_2(\varphi+1,a+1,b+1;c+1,d+1;1)\frac{(a)(b)}{(c)(d)}$$
$$+(1-\alpha)\ {}_3\mathrm{F}_2(\varphi,a,b;c,d;1)\leq 0. \tag{2.14}$$

Proof. If we set $k=1$ in (2.12), the proof is completed.

Corollary 8. $\mathrm{R}_1(\varphi,a;c;k;z)=z[2-{}_2\mathrm{R}_1(\varphi,a;c;k;z)]$ is in the class $C(\lambda,\alpha)$ if and only if

$$(1-\lambda\alpha)_2\mathrm{R}_1(\varphi+2,a+2k;c+2k;k;1)\frac{(\varphi+1)\Gamma(a+2k)}{\Gamma(c+2k)}$$
$$+(3-\alpha-2\lambda\alpha)_2\mathrm{R}_1(\varphi+1,a+k;c+k;k;1)\frac{\Gamma(a+k)}{\varphi\Gamma(c+k)}$$
$$+(1-\alpha)\ {}_2\mathrm{R}_1(\varphi,a;c;1)\frac{\Gamma(a)}{\varphi\Gamma(c)}\leq 0. \tag{2.15}$$

Proof. If we set $b=d$ in (2.12), the proof is completed.

3.An Integral Operator

In this section, we obtain similar results (those obtained in the preceding section) for a particular integral operator $\mathrm{G}(\varphi,a,\mathrm{b};c,d;k;z)$ that acts on ${}_3\mathrm{R}_2(\varphi,a,\mathrm{b};c,d;k;z)$ we define:

$$\mathrm{G}(\varphi,a,\mathrm{b};c,d;k;z)=\int_0^z {}_3\mathrm{R}_2(\varphi,a,\mathrm{b};c,d;k;x)dx \tag{3.1}$$

Theorem 3.1. Let $k\in R, k>0$. Then $\mathrm{G}(\varphi,a,b;c,d;k;z)$ is in the class $T(\lambda,\alpha)$, if and only if

$$(1-\lambda\alpha)_3\mathrm{R}_2(\varphi,a,\mathrm{b};c,d;k;1)\frac{\Gamma(a)\Gamma(\mathrm{b})\Gamma(\mathrm{c+k})\Gamma(\mathrm{d+k})}{(\varphi)\Gamma(\mathrm{c})\Gamma(\mathrm{d})\Gamma(\mathrm{a+k})\Gamma(\mathrm{b+k})}$$
$$-\alpha(1-\lambda)\,{}_3\mathrm{R}_2(\varphi-1,a-k,\mathrm{b-k};c-k,d-k;k;1)\frac{\Gamma(a-k)\Gamma(\mathrm{b-k})\Gamma(\mathrm{c+k})\Gamma(\mathrm{d+k})}{(\varphi-1)_2\Gamma(\mathrm{c-k})\Gamma(\mathrm{d-k})\Gamma(\mathrm{a+k})\Gamma(\mathrm{b+k})}$$
$$+\alpha(1-\lambda)\left[\frac{\Gamma(a-k)\Gamma(\mathrm{b-k})\Gamma(\mathrm{c+k})\Gamma(\mathrm{d+k})}{(\varphi-1)_2\Gamma(\mathrm{c-k})\Gamma(\mathrm{d-k})\Gamma(\mathrm{a+k})\Gamma(\mathrm{b+k})}\right]\leq 0. \tag{3.2}$$

Proof. Since

$$\mathrm{G}(\varphi,a,b;c,d;k;z)=z-\left|\frac{(\varphi)\Gamma(\mathrm{c})\Gamma(\mathrm{d})}{\Gamma(a)\Gamma(b)}\right|\sum_{n=2}^{\infty}\frac{(\varphi+1)_{n-2}\Gamma(a+k+k(n-2))\Gamma(b+k+k(n-2))}{\Gamma(c+k+k(n-2))\Gamma(d+k+k(n-2))}\frac{z^n}{(1)_n} \tag{3.3}$$

By Lemma 1, it is sufficient to prove that

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(\varphi+1)_{n-2}\Gamma\big(a+k+k(n-2)\big)\Gamma\big(b+k+k(n-2)\big)\Gamma(c+k)\Gamma(d+k)}{\Gamma(c+k+k(n-2))\Gamma(d+k+k(n-2))\Gamma(a+k)\Gamma(b+k)(1)_n}$$
$$\leq\left|\frac{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)}{\Gamma(a+k)\Gamma(b+k)\Gamma(c)\Gamma(d)}\right|(1-\alpha)$$

Now we have

$$\sum_{n=2}^{\infty}[n(1-\lambda\alpha)-\alpha(1-\lambda)]\ \frac{(\varphi+1)_{n-2}\Gamma\big(a+k+k(n-2)\big)\Gamma\big(b+k+k(n-2)\big)\Gamma(c+k)\Gamma(d+k)}{\Gamma(c+k+k(n-2))\Gamma(d+k+k(n-2))\Gamma(a+k)\Gamma(b+k)(1)_n}$$
$$=(1-\lambda\alpha)\sum_{n=0}^{\infty}(n+2)\frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)\Gamma(c+k)\Gamma(d+k)}{\Gamma(c+k+kn)\Gamma(d+k+kn)\Gamma(a+k)\Gamma(b+k)(1)_{n+2}}$$
$$-\alpha(1-\lambda)\sum_{n=0}^{\infty}\frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)\Gamma(c+k)\Gamma(d+k)}{\Gamma(c+k+kn)\Gamma(d+k+kn)\Gamma(a+k)\Gamma(b+k)(1)_{n+2}}$$

i.e.
$$=(1-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(\varphi+1)_n\Gamma(a+k+kn)\Gamma(b+k+kn)\Gamma(c+k)\Gamma(d+k)}{\Gamma(c+k+kn)\Gamma(d+k+kn)\Gamma(a+k)\Gamma(b+k)(1)_{n+1}}$$
$$-\alpha(1-\lambda)\sum_{n=1}^{\infty}\frac{(\varphi)_n\Gamma(a+kn)\Gamma(b+kn)\Gamma(c+k)\Gamma(d+k)}{(\varphi)\Gamma(c+kn)\Gamma(d+kn)\Gamma(a+k)\Gamma(b+k)(1)_{n+1}}$$

on simplification (details are avoided) we get

$$= (1-\lambda\alpha)\left[{}_3\mathrm{R}_2(\varphi,a,b;c,d;k;1)\frac{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)}{(\varphi)\Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)} - \frac{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)}{(\varphi)\Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)}\right]$$
$$-\alpha(1-\lambda)\,{}_3\mathrm{R}_2(\varphi-1,a-k,b-k;c-k,d-k;k;1)\frac{\Gamma(a-k)\Gamma(b-k)\Gamma(c+k)\Gamma(d+k)}{(\varphi-1)_2\Gamma(c-k)\Gamma(d-k)\Gamma(a+k)\Gamma(b+k)}$$
$$+\alpha(1-\lambda)\left[\frac{\Gamma(a-k)\Gamma(b-k)\Gamma(c+k)\Gamma(d+k)}{(\varphi-1)_2\Gamma(c-k)\Gamma(d-k)\Gamma(a+k)\Gamma(b+k)} + \frac{(\varphi-1)\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)}{(\varphi-1)_2\Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)}\right].$$

The proof is completed.

Corollary 9. $G(\varphi,a,b;c,d;z)$ is in the class $T(\lambda,\alpha)$, if and only if

$$(1-\lambda\alpha)\,{}_3\mathrm{F}_2(\varphi,\mathrm{a},\mathrm{b};c,d;1)\frac{(\mathrm{c})(\mathrm{d})}{(\varphi)(\mathrm{a})(\mathrm{b})}$$
$$-\alpha(1-\lambda)\,{}_3\mathrm{F}_2(\varphi-1,\mathrm{a}-1,\mathrm{b}-1;c-1,d-1;1)\frac{(\mathrm{c}-1)_2(\mathrm{d}-1)_2}{(\varphi-1)_2(\mathrm{a}-1)_2(\mathrm{b}-1)_2}$$
$$+\alpha(1-\lambda)\left[\frac{(\mathrm{c}-1)_2(\mathrm{d}-1)_2}{(\varphi-1)_2(\mathrm{a}-1)_2(\mathrm{b}-1)_2}\right] \le 0. \tag{3.4}$$

Proof. If we set k =1 in (3.2), the proof is completed.

Corollary 10. Let $k \in R, k > 0$ then $G(\varphi,a;c;k;z)$ is in the class $T(\lambda,\alpha)$, if and only if

$$(1-\lambda\alpha)\,{}_2\mathrm{R}_1(\varphi,a;c;k;1)\frac{\Gamma(a)\Gamma(c+k)}{(\varphi)\Gamma(c)\Gamma(a+k)}$$
$$-\alpha(1-\lambda)\,{}_2\mathrm{R}_1(\varphi-1,a-k;c-k;k;1)\frac{\Gamma(a-k)\Gamma(c+k)}{(\varphi-1)_2\Gamma(c-k)\Gamma(a+k)}$$
$$+\alpha(1-\lambda)\left[\frac{\Gamma(a-k)\Gamma(c+k)}{(\varphi-1)_2\Gamma(c-k)\Gamma(a+k)}\right] \le 0. \tag{3.5}$$

Proof. If we set $b = d$ in (3.2), the proof is completed.

Remark 1. We observe that $G(\varphi,a,b;c,d;z) \in C(\lambda,\alpha)$ if and only if $z\ {}_3\mathrm{R}_2(\varphi,a,b;c,d;k;z) \in T(\lambda,\alpha)$. Thus any result of functions belonging to the class $T(\lambda,\alpha)$ about $z\ {}_3\mathrm{R}_2(\varphi,a,b;c,d;k;z)$ leads to that of functions belonging to the class $C(\lambda,\alpha)$. Hence we obtain the following analogous result to Theorem 2.1.

Theorem 3.2. $G(\varphi,a,b;c,d;z)$ defined by (3.1) is in $C(\lambda,\alpha)$ if and only if

$$(1-\lambda\alpha)\,{}_3\mathrm{R}_2(\varphi+1,a+k,b+k;c+k,d+k;k;1)$$
$$+(1-\alpha)\,{}_3\mathrm{R}_2(\varphi,a,b;c,d;k;1)\frac{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)}{(\varphi)\Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)} \le 0. \tag{3.6}$$

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