On The Solution of Fractional Kinetic Equation

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Research Article

Abstract: The object of this article is to study the fractional kinetic equation which deals with certain problems in science and engineering. The results are obtained in a compact form containing Mittag-Leffler function, Robotov's function and Hypergeometric function.

Keywords: Kinetic equation; Fractional operators; Mittag-Leffler function; Laplace transform; Hypergeometric function.

1.Introduction

The fractional kinetic equation have importance in certain field of Physical phenomena governing diffusion in porous media, reaction and relaxtion processes in complex systems and anomalous diffusion etc. For detail , one can see the monographs by Hilfer[10], Kilbas et al.[12], Kiryakova[13] and Podlubny[17] . Fractional kinetic equations are studied by Hille and Tamarkin[11], Glöckle and Nonnenmacher[7], Saichev and Zaslavsky[20], Saxena et al.[21-23] and Zaslavsky[28].

The special function of the form

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\quad \mathrm{Re}(\alpha)>0, z\in C,$$

(1.1)

was introduced by Mittag-Leffler[15], in 1903.

In 1905, this function was generalized by Wiman [27] as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, z\in C.$$

(1.2)

Both functions are entire function of order $\rho = 1/\alpha$ and type $\sigma = 1$. The classical theory of these functions is presented in the handbook by Erdilyi et al.[6, section 18.1], while recent results are given in the Dzherbashyan[2,3]. Recently the interest to these functions has grown up by their application in some evolution problems[8].

Prabhakar[18] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ as

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k z^k}{\Gamma(\alpha k+\beta)k!},\ \mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma)>0, z\in C,$$

(1.3)

where $(\gamma)_k$ is Pochhammer symbol [5, section 2.1.1] ,

$(\gamma)_k = \gamma(\gamma+1)\ldots\ldots\ldots\ldots(\gamma+k-1)$, (k = 1,2,3……)

and $(\gamma)_0 = 1$.

In 2007, Shukla and Prajapati [25] introduced the formula $E^{\gamma,q}_{\alpha,\beta}(z)$ which is defined for $z,\alpha,\beta,\gamma \in C$; Re(α) , Re(β) , Re(γ) > 0 and q ∈ (0,1) ∪ N as :

$$E^{\gamma,q}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_{kq} z^k}{\Gamma(\alpha k+\beta)k!}, \tag{1.4}$$

where $(\gamma)_{kq} = \frac{\Gamma(\gamma+kq)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol[19].

The Mellin – Ross function [14] is defined as

$$E_t(v,a)=t^v\sum_{k=0}^{\infty}\frac{(at)^k}{\Gamma(k+v+1)}. \tag{1.5}$$

Operators of fractional calculus have been studied by Srivastava and Saxena [26] . Due to their application in solution of integral and differintegral equation , the Riemann- Liouville operator of fractional calculus is defined by[14,16].

$${}_0D_t^{-\alpha}\,\mathrm{N(t)} = \frac{1}{\Gamma(\alpha)}\int_0^t N(u)\ (t-u)^{\alpha-1}\ du,$$

Re(α) >0 , (1.6)

with ${}_0D_t^{0}$ N(t) = N(t) , and

$${}_0D_t^{\alpha}\,N(t) = \frac{d^n}{dt^n}\ {}_0D_t^{n-\alpha}\,\mathrm{N(t)};\quad \mathrm{Re}(\alpha), \mathrm{Re}(n-\alpha) > 0. \tag{1.7}$$

By using (1.6) and (1.7) , it yields that for N(t) = t^ρ, we have

$${}_0D_t^{-\alpha}\,t^\rho = \frac{\Gamma(\rho+1)}{\Gamma\rho+\alpha+1)}\,t^{\rho+\alpha};\ \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\rho)>-1, t>0, \tag{1.8}$$

and

$${}_0D_t^{\alpha}\,t^\rho = \frac{\Gamma(\rho+1)}{\Gamma\rho-\alpha+1)}\,t^{\rho-\alpha},\ 0 < \mathrm{Re}(\alpha) <1, \mathrm{Re}(\rho)>-1, t>0. \tag{1.9}$$

The standard kinetic equation ,

$$\frac{d}{dt}N_i\,(t) = -\,c_i\,N_i\,(t);\qquad c_i >0. \tag{1.10}$$

By integrating standard kinetic equation , we obtain [9,p.58] ,

$$N_i\,(t) - N_i\,(0) = -\,c_i\,D_t^{-1}\,N_i\,(t), \tag{1.11}$$

where D_t^{-1} is the standard Riemann integral operator. Here N_i (t) is the number density of species i, which is function of time (t) and N_i (t=0)= N_0 is the number density of species i at time t=0, [9].

The generalize form of (1.11) by dropping index i , we obtain ,

$$\mathrm{N(t)} - N_0 = -\,c_i\,D_t^{-v}\,\mathrm{N(t)}. \tag{1.12}$$

We can obtain more general integral equation than (1.12) by Laplace transform method by invoking the result given by Erdelyi et al. [4,p.182],

$$\mathcal{L}\{D_t^{-v}\} = s^{-v}\,\mathrm{F(s)}, \tag{1.13}$$

where

$\mathcal{L}\{f(t)\}$ = F(s) = $\int_0^\infty e^{-su}$ f(u) du , Re(s)>0. (1.14)

Further, we use a fractional calculus method to obtain solution of differintegral equations related to fractional kinetic equation (cf. [1]).

Our paper is devoted to further investigation of the fractional kinetic equation (1.11). The results are derived in a compact form by the application of Laplace transforms, which are suitable for numerical computation.

2. Laplace transform method:

The integral equation,

N(t) - N_0 f(t) = - $c^v\, D_t^{-v}$ N (t), (2.1)

it follows from (1.12) that ,

$\mathcal{L}\{N(t)\}$ - N_0 F(s) = - $c^v\, s^{-v}\, \mathcal{L}\{N(t)\}$,

which reduces to

$$\mathcal{L}\{N(t)\} = N_0 \frac{F(S)}{(1+C^v S^{-v})},$$

provided s>c , we obtain the power series expansion as

$$\mathcal{L}\{N(t)\} = N_0 \sum_{m=0}^{\infty} (-1)^m (c)^{mv} (s)^{-mv} F(s)$$

now we apply inverse Laplace transform, then

$$\mathrm{N(t)} = N_0 \sum_{m=0}^{\infty} \frac{(-1)^m (c)^{mv}}{\Gamma(mv)} t^{mv-1} * f(t) \qquad (2.2)$$

where

$$(\mathrm{t})^{mv-1} * \mathrm{f(t)} = \int_0^t (t-u)^{mv-1} \mathrm{f(u)\, du}\ ; \qquad (2.3)$$

is convolution integral.

Main Results :

A number of cases follows from (2.2) :

(I) If f(t) = $t^{\mu-1}\ {}_1F_0$ (μ ; _ ; -at), then we can write convolution integral (2.3) as

$$(\mathrm{t})^{mv-1} * t^{\mu-1}\ {}_1F_0(\mu\,;\,\_\,;-at) = \int_0^t (t-u)^{mv-1} u^{\mu-1}\ {}_1F_0(\mu\,;\,\_\,;-at)\, du$$

$$= \int_0^t (t-u)^{mv-1} u^{\mu-1} \sum_{k=0}^{\infty} \frac{(\mu)_k (-at)^k}{k!},$$

$$= \sum_{k=0}^{\infty} \frac{(\mu)_k (-at)^k}{k!} \int_0^t \mathrm{t}^{mv-1} \left(1-\frac{u}{t}\right)^{mv-1} u^{\mu+k-1} du,$$

and substituting $x = {}^{u}/_{t}$, which yields

$$= \sum_{k=0}^{\infty} \frac{(\mu)_k (-a)^k}{k!} \mathrm{t}^{mv+\mu+k-1} \int_0^1 (1-x)^{mv-1} x^{\mu+k-1} dx,$$

$$= \sum_{k=0}^{\infty} \frac{(\mu)_k (-a)^k}{k!} \mathrm{t}^{mv+\mu+k-1} \frac{\Gamma(mv)\,\Gamma(\mu+k)}{\Gamma(mv+\mu+k)}$$

The simplification of above equation gives ,

$$= \frac{\Gamma(\mu)\,\Gamma(mv)}{\Gamma(mv+\mu)} t^{mv+k-1} \sum_{k=0}^{\infty} \frac{(\mu)_k (\mu)_k (-at)^k}{(mv+\mu)_k k!},$$

$$= \frac{\Gamma(\mu)\,\Gamma(mv)}{\Gamma(mv+\mu)} \mathrm{t}^{mv+k-1}\ {}_2F_1(\mu\,,\mu\,;mv+\mu\,;-at). \qquad (2.4)$$

and by using (2.4) , integral equation (2.2) becomes ,

$$\mathrm{N(t)} = N_0\, \Gamma(\mu)\, \mathrm{t}^{\mu-1} \sum_{m=0}^{\infty} \frac{(-1)^m (c)^{mv}}{\Gamma(mv+\mu)}\ {}_2F_1(\mu\,,\mu\,;mv+\mu\,;-at),$$

$$\mathrm{N(t)} = N_0\, \Gamma(\mu)\, \mathrm{t}^{\mu-1} E_{v,\mu}(-c^v t^v)\ {}_2F_1(\mu\,,\mu\,;mv+\mu\,;-at). \qquad (2.5)$$

where , Re(a) , Re(mv+μ) > 0.

(II) If f(t) = $E_t(v,a) = t^v \sum_{p=0}^{\infty} \frac{(at)^p}{\Gamma(p+v+1)}$, (2.6)

where $E_t(v,a)$ is Mellin Ross function defined by (1.5).

Then convolution integral equation (2.3) written as ,

$$\mathrm{t}^{mv-1} * E_t(v,a) = \int_0^t (t-u)^{mv-1} E_u(v,a)\, du,$$

$$= \int_0^t (t-u)^{mv-1} u^v \sum_{p=0}^{\infty} \frac{(au)^p}{\Gamma(p+v+1)},$$

$$= \sum_{p=0}^{\infty} \frac{(a)^p}{\Gamma(p+v+1)} \int_0^t (t-u)^{mv-1} \mathrm{u}^{p+v} du,$$

$$= \sum_{p=0}^{\infty} \frac{(a)^p}{\Gamma(p+v+1)} \int_0^t \mathrm{t}^{mv-1} \left(1-\frac{u}{t}\right)^{mv-1} u^{p+v} du,$$

substituting $x = {}^{u}/_{t}$, we obtained

$$= \sum_{p=0}^{\infty} \frac{(a)^p}{\Gamma(p+v+1)} t^{mv+p+v} \int_0^1 (1-x)^{mv-1} x^{p+v} dx,$$

$$= \sum_{p=0}^{\infty} \frac{(a)^p}{\Gamma(p+v+1)} t^{mv+p+v} \int_0^1 (1-x)^{mv-1} x^{(p+v+1)-1} dx$$

on simplification of above equation ,

$$t^{mv-1} * E_t(v,a) = \sum_{m=0}^{\infty} \frac{(-ac^v)^m t^{m(v+1)}}{\Gamma(m+1)(v+1)} t^{mv+p+v} \qquad (2.7)$$

Now by using equation (2.7) in (2.2) , it becomes

$$N(t) = N_0 \sum_{p,m=0}^{\infty} \frac{(-1)^m c^{mv} (a)^p t^{mv+p+v}}{\Gamma(mv+p+v+1)}$$

If we assume that $p = m$, then

$$N(t) = N_0\, t^v \sum_{m=0}^{\infty} \frac{(-ac^v)^m t^{m(v+1)}}{\Gamma(m+1)(v+1)}$$

$$N(t) = N_0 R_v(-ac^v, t)\ . \qquad (2.8)$$

where is $R_v(-ac^v, t)$ Robotov's function [14].

We can write (2.8) in term of Mittag – Leffler function as ,

$N(t) = N_0\, t^v\, E_{v+1,v+1}(-ac^v, t^{v+1})$.

3. Special Case:

By (2.5) , we have

$$\mathrm{N(t)} = N_0\, \Gamma(\mu)\, \mathrm{t}^{\mu-1} E_{v,\mu}(-c^v t^v)\ {}_2F_1(\mu\,,\mu\,;mv+\mu\,;-at)$$

N(t) = $N_0\,\Gamma(\mu)\, t^{\mu-1}\, E_{v,\mu}(-c^v t^v)$

$$\sum_{k=0}^{\infty}\frac{(\mu)_k(\mu)_k(-at)^k}{(mv+\mu)_k\,k!},$$

if we set k = 0 in above result , then we have

$$N(t)=N_0\Gamma(\mu)t^{\mu-1}E_{v,\mu}(-c^v t^v).$$

which is well known result given by Saxena and Kalla[24].

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