An Oscillatory MHD Visco-Elastic Fluid Flow through a Porous Medium Bounded by Rotating Porous Channel in the presence of Hall Current

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Research Article

Abstract: A theoretical solution for an oscillatory flow of an electrically conducting, viscous, incompressible visco-elastic fluid is obtained without neglecting Hall current. The magnetohydrodynamic (MHD) flow is bounded by two infinite horizontal plates filled with porous medium. The fluid is injected with constant velocity through the lower stationary plate and the upper plate is subjected to the same constant suction velocity. The effects of Hall current and Hartmann numbers on velocity profile and shear stress for different values of the visco-elastic parameter with the combination of the other flow parameters are illustrated graphically and physical aspects of the problem are discussed.

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1. Introduction

The study of flows through porous medium have stimulated considerable interest due its applications in the fields of agricultural engineering for irrigation processes; in petroleum technology to study petroleum transport; in chemical engineering for filtration and purification processes. Raptis [1] investigated the unsteady two-dimensional flow through a porous medium bounded by an infinite porous plate subjected to a constant suction and variable temperature. Further, Raptis and Perdikis [2] studied the problem of free convective flow through a porous medium bounded by a vertical porous plate with constant suction when the free stream velocity oscillates in time about a constant mean value. Singh et al. [3] investigated the problem of heat transfer in three dimensional flow through a porous medium with periodic permeability. Singh and Verma [4] studied further the three dimensional oscillatory flow through a porous medium where the free stream velocity oscillates in time about a non-zero constant mean. Singh et al. [5] studied the effect of permeability variation on the heat transfer and three dimensional flow through a highly porous medium bounded by an infinite porous plate with constant suction. Attia [6] studied the Hall current effect on the velocity and temperature fields of an unsteady Hartmann number. Singh and Mathew [7] studied the injection/suction effect on a hydromagnetic oscillatory flow in a horizontal porous channel in a rotating system. Choudhury and Das [8] extended the problem studied by Singh and Mathew [7] to the case of visco-elastic fluid. Singh and Kumar [9] investigated the problem of on oscillatory MHD flow through a porous medium bounded by rotating porous channel in the presence of Hall current. The aim of the present investigation is to study the effect of the Hall current on the visco-elastic fluid flow when the porous horizontal channel filled with a porous medium is rotating about an axis normal to the planes of the plates.

2. Mathematical Analysis

Consider an unsteady oscillatory flow of an electrically conducting visco-elastic, incompressible second order fluid through a porous medium bounded between two insulated infinite parallel porous plates distance d apart. The fluid is injected with constant velocity w_0 through the lower stationary plate and is being sucked with the same velocity w_0 through the upper plate which is oscillating in its own plane with a velocity $U^*(t^*)$ about a non-zero constant mean velocity U_0 . A coordinate system is taken with the origin at the lower stationary plate lying in $x^* - y^*$ plane and x^*-axis parallel to the direction of motion of the upper plate. The z^* -axis taken perpendicular to the planes of the plates, is the axis of the rotation about which the entire system is rotating with constant angular velocity Ω^*. A strong magnetic field of uniform strength H_0 is applied along z^*- axis. The magnetic Reynolds number is considered to be small so that the induced magnetic field is neglected. Since the plates are infinite in extent, all the physical quantities except the pressure depend only on z^* and t^* for this fully developed laminar flow.

The constitutive equation for the incompressible second order fluid is of the form

$$\sigma = -pI + \mu_1 A_1 + \mu_2 A_2 + \mu_3 (A_1)^2 \tag{1}$$

where σ is the stress tensor, A_n (n=1, 2) are the kinematic Rivlin-Ericksen tensors; μ_1, μ_2, μ_3 are the material coefficients describing the viscosity, elasticity and cross-viscosity respectively. The material coefficients μ_1, μ_2, μ_3 are taken constants with μ_1 and μ_3 as positive and μ_2 as negative (Coleman and Markovitz [10]). The equation (1) was derived by Coleman and Noll [11] from that of the simple fluids by assuming that stress is more sensitive to the recent deformation than to the deformation that occurred in the distant past.

Denoting the velocity components u^*, v^*, w^* in the x^*, y^*, z^* directions, respectively, the continuity equation $\frac{\partial w^*}{\partial z^*} = 0$ gives on integration $w^* = w_0$ and solenoidal relation for the magnetic field $\nabla . \vec{H} = 0$ gives $H_z^* = H_0$ (constant) everywhere in the flow field. The physical configuration of the problem is shown in Figure 1.

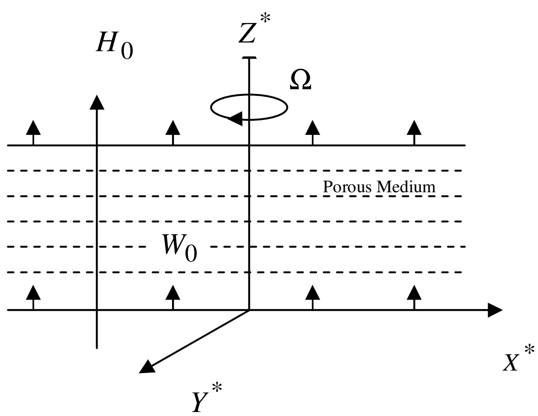

Figure1: *Physical configuration of the problem.*

The equation of conservation of electric charge $\nabla . \vec{J} = 0$ gives J_z^*=constant. This constant is zero i.e. $J_z^* = 0$ at the plates which are electrically non-conducting. Taking Hall current into account the generalized Ohm's law (Cowling [12]) is of the form

$$\vec{J} + \frac{\omega_e \tau_e}{H_0} \vec{J} \times \vec{H} = \sigma\left(\vec{E} + \mu_e \vec{V} \times \vec{H}\right), \qquad (2)$$

where $\vec{V}$ is the velocity vector, $\vec{H}$ is the magnetic field, $\vec{J}$ is the current density, $\vec{E}$ is the electric field, σ is the electric conductivity, μ_e is the magnetic permeability, ω_e is the cyclotron frequency and τ_e is the electron collision time.

For the x^* and y^* components of Ohm's law (2), for large magnetic field, which include Hall current, are

$$J_x^* + \omega_e \tau_e J_y^* = \sigma\left(E_x^* + \mu_e H_0 v^*\right),$$

$$J_y^* + \omega_e \tau_e J_x^* = \sigma\left(E_y^* - \mu_e H_0 u^*\right).$$

Since the external electric field arising due to polarization of charges is negligible.

Hence $E_x^* = E_y^* = 0.$ Therefore, solving for J_x^* and J_y^*, we get

$$J_x^* = \frac{\sigma \mu_e H_0 (mu^* + v^*)}{(1+m^2)} \quad \text{and} \quad J_y^* = \frac{\sigma \mu_e H_0 (mv^* - u^*)}{(1+m^2)}.$$

Under the above assumptions, the governing equations for the flow, in presence of Hall current, are as follows:

$$\frac{\partial u^*}{\partial t^*} + w_0 \frac{\partial u^*}{\partial z^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \nu_1 \frac{\partial^2 u^*}{\partial z^{*2}} + \nu_2\left(\frac{\partial^3 u^*}{\partial z^{*2}\partial t^*} + w_0 \frac{\partial^3 u^*}{\partial z^{*3}}\right) + 2\Omega^* v^* + \frac{\sigma H_0^2 (mv^* - u^*)}{\rho(1+m^2)} - \nu_1 \frac{u^*}{k^*}, \qquad (3)$$

$$\frac{\partial v^*}{\partial t^*} + w_0 \frac{\partial v^*}{\partial z^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial y^*} + \nu_1 \frac{\partial^2 v^*}{\partial z^{*2}} + \nu_2\left(\frac{\partial^3 v^*}{\partial z^{*2}\partial t^*} + w_0 \frac{\partial^3 v^*}{\partial z^{*3}}\right) - 2\Omega^* u^* - \frac{\sigma H_0^2 (mu^* + v^*)}{\rho(1+m^2)} - \nu_1 \frac{v^*}{k^*} \qquad (4)$$

where $\nu_i = \frac{\mu_i}{\rho}$, $i = 1,2$ are the kinematic viscosity, t^* is the time, ρ is the density, p^* is the pressure and k^* is the permeability of the porous medium, $m = \omega_e \tau_e$ is the Hall parameter.

The boundary conditions for the problem are

$u^* = v^* = 0, w^* = w_0$ at $z^* = 0$,

$u^* = U^*(t) = U_0(1 + \varepsilon \cos \omega^* t^*), v^* = 0, w^* = w_0$ at $z^* = d$ (5)

where ω^* is the frequency of oscillations, U_0 is the mean velocity and ε is a very small positive constant.

Eliminating the pressure gradient, under the usual boundary layer approximations, equations (3) and (4) reduces to

$$u_t^* + w_0 u_z^* = \nu_1 u_{zz}^* + U_t^* + 2\Omega^* v^* + \nu_2 (w_0 u_{zzz}^* + u_{zzt}^*) + \frac{\sigma H_0^2 (mv^* - u^* + U^*)}{\rho(1+m^2)} - \frac{\nu_1}{k^*}(u^* - U^*) \qquad (6)$$

$$v_t^* + w_0 v_z^* = \nu_1 v_{zz}^* - 2\Omega^*(u^* - U^*)$$

$$+\nu_2(w_0 v^*_{zzz} + v^*_{zzt}) + \frac{\sigma H_0^2(mu^* + v^* - mU^*)}{\rho(1+m^2)}$$
$$-\nu_1 \frac{v^*}{k^*} \tag{7}$$

Introducing the following non-dimensional quantities $\eta = \frac{z^*}{d}$, $t = \omega^* t^*$, $u = \frac{u^*}{U_0}$, $v = \frac{v^*}{U_0}$, $\Omega = \Omega^* \frac{d^2}{\nu_1}$, the rotation parameter, $\omega = \omega^* \frac{d^2}{\nu_1}$, the frequency parameter, $s = \frac{\omega_0 d}{\nu_1}$ is the injection/suction parameter, $k = \frac{k^*}{d^2}$ is the permeability parameter, $M = H_0 d\sqrt{\frac{\sigma}{\mu}}$ is the Hartmann number, and $U = \frac{U^*}{U_0}$ into the equations (6) and (7), we get

$$\omega u_t + su_\eta = u_{\eta\eta} + \omega U_t + 2\Omega v + \frac{s\nu_2}{d^2} u_{\eta\eta\eta}$$
$$+\frac{\nu_2}{d^2} u_{\eta\eta t} + \frac{M^2(mv - u + v)}{(1+m^2)} - \frac{u-U}{k} \tag{8}$$

$$\omega v_t + sv_\eta = v_{\eta\eta} - 2\Omega(u - U) + \frac{s\nu_2}{d^2} v_{\eta\eta\eta}$$
$$+\frac{\nu_2}{d^2} v_{\eta\eta t} + \frac{M^2(mv - u + v)}{(1+m^2)} - \frac{v}{k} \tag{9}$$

subject to boundary conditions

$$u = v = 0 \quad \text{at } \eta = 0,$$
$$u = U(t) = 1 + \varepsilon \cos t,\ v = 0 \quad \text{at } \eta = 1 \tag{10}$$

Equations (8) and (9) can now be combined into a single equation, by introducing the complex function $q = u + iv$, as

$$\omega q_t + sq_\eta = q_{\eta\eta} + \omega U_t - \frac{q-U}{k} + \alpha(sq_{\eta\eta\eta}$$
$$+ q_{\eta\eta t}) - \lambda(q - U), \tag{11}$$

subject to the boundary conditions

$$q = 0 \quad \text{at } \eta = 0$$
$$q = U(t) = 1 + \frac{\varepsilon}{2}(e^{it} + e^{-it}) \ \text{at } \eta = 1 \tag{12}$$

where $\alpha = \frac{\mu_2}{\rho d^2}$, the visco-elastic parameter and

$$S = \left[2\Omega i + \frac{M^2(1+im)}{(1+m^2)} + \frac{1}{k}\right].$$

3. Solution of the problem

In order to solve equation (11) subject to the boundary conditions (12), we look for a solution of the form

$$q(\eta, t) = q_0(\eta) + \frac{\varepsilon}{2}\{q_1(\eta)e^{it} + q_2(\eta)e^{-it}\} \tag{13}$$

Substituting (13) into the equations (11) and (12) and comparing the harmonic and non-harmonic terms, we get

$$\alpha s q_0''' + q_0'' - sq_0' - \lambda q_0 = -\lambda, \tag{14}$$
$$\alpha s q_1''' + (i\alpha + 1)q_1'' - sq_1' - (\lambda + i\omega)q_1 = -(\lambda + i\omega) \tag{15}$$
$$\alpha s q_2''' + (-i\alpha + 1)q_2'' - sq_2' - (\lambda - i\omega)q_2 = -(\lambda - i\omega) \tag{16}$$

where primes denote differentiation with respect to η.

The corresponding transformed boundary conditions are

$$q_0 = q_1 = q_2 = 0 \quad \text{at} \quad \eta = 0,$$
$$q_0 = q_1 = q_2 = 1 \quad \text{at} \quad \eta = 1 \tag{17}$$

To solve the equations (14) to (16) under boundary conditions (17), we consider very small value of non-Newtonian parameter α, and substituting

$$q_0(\eta) = q_{00}(\eta) + \alpha q_{01} + o(\alpha^2),$$
$$q_1(\eta) = q_{10}(\eta) + \alpha q_{11} + o(\alpha^2),$$
$$q_2(\eta) = q_{20}(\eta) + \alpha q_{21} + o(\alpha^2) \tag{18}$$

into equations (14) to (16) and boundary conditions (18) up to first order of α and equating the coefficients of like powers of α, we obtain the following sets of ordinary differential equations and corresponding boundary conditions:

$$\left.\begin{aligned} &q_{00}'' - sq_{00}' - \lambda q_{00} = -\lambda \\ &sq_{00}''' + q_{01}'' - sq_{01}' - \lambda q_{01} = 0 \end{aligned}\right\} \tag{19}$$

with

$$\begin{aligned} &q_{00} = q_{01} = 0 \quad \text{at} \quad \eta = 0, \\ &q_{00} = 1,\ q_{01} = 0 \quad \text{at} \quad \eta = 1 \end{aligned} \tag{20}$$

$$\left.\begin{aligned} &q_{10}'' - sq_{10}' - (\lambda + i\omega)q_{10} = -(\lambda + i\omega) \\ &sq_{10}''' + iq_{10}'' + q_{11}'' - sq_{11}' - (\lambda + i\omega)q_{11} = 0 \end{aligned}\right\} \tag{21}$$

with

$$\begin{aligned} &q_{10} = 0,\ q_{11} = 0 \quad \text{at } \eta = 0, \\ &q_{10} = 1,\ q_{11} = 0 \quad \text{at } \eta = 1 \end{aligned} \tag{22}$$

$$\left.\begin{aligned} &q_{20}'' - sq_{20}' - (\lambda - i\omega)q_{20} = -(\lambda - i\omega) \\ &sq_{20}''' - iq_{20}'' + q_{21}'' - sq_{21}' - (\lambda - i\omega)q_{21} = 0 \end{aligned}\right\} \tag{23}$$

with

$$\begin{aligned} &q_{20} = 0,\ q_{21} = 0 \quad \text{at } \eta = 0, \\ &q_{20} = 1,\ q_{21} = 0 \quad \text{at } \eta = 1 \end{aligned} \tag{24}$$

Solving the equations (19), (21), (23) under the boundary conditions (20), (22), (24) respectively and substituting these values in (18), we get the solutions for velocity. The solutions and constants of the differential equations are obtained but not presented here for the sake of brevity.

4. Results and discussion

Now for the resultant velocities and the shear stress of the steady and unsteady flow, we write

$$u_0(\eta)+i\,v_0(\eta)=q_0(\eta) \tag{25}$$

and

$$u_1(\eta)+i\,v_1(\eta)=q_1(\eta)\,e^{it}+q_2(\eta)\,e^{-it} \tag{26}$$

The steady part consists of u_0 as the primary and v_0 as the secondary velocity components. The amplitude and phase difference due to these primary and secondary velocities for the steady flow are given by

$$R_0=\sqrt{u_0^2+v_0^2}\ ,\quad \theta_0=\tan^{-1}\left(\frac{v_0}{u_0}\right) \tag{27}$$

Figures 2 and 3 depict the resultant velocity R_0 for the steady and unsteady part of the flow against η to observe the visco-elastic effects for the various values of Hall current (m) and Hartmann number (M). It is observed from figures 2 and 3 that both R_0 and R_1 increase rapidly

m	M	α	
1	2	0	*I*
1	2	−0.05	*II*
1	2	−0.10	*III*
4	2	0	*IV*
4	2	−0.05	*VI*
4	2	−0.10	*VI*
1	4	0	*VII*
1	4	−0.05	*VIII*
1	4	−0.10	*IX*

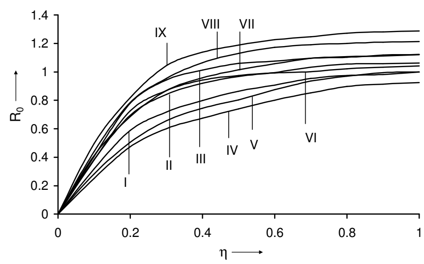

Figure 2: *Variation of resultant velocity* R_0 *against* η *with* $s=0.3,\ \Omega=1,\ k=0.2.$

m	M	α	
1	2	0	*I*
1	2	−0.05	*II*
1	2	−0.10	*III*
4	2	0	*IV*
4	2	−0.05	*VI*
4	2	−0.10	*VI*
1	4	0	*VII*
1	4	−0.05	*VIII*
1	4	−0.10	*IX*

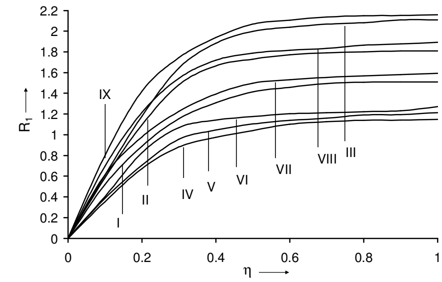

Figure 3: *Variation of resultant velocity* R_1 *against* η *with* $s=0.3,\ \Omega=1,\ k=0.2,\ \omega=5.$

from zero near the stationary plate for Newtonian $(\alpha=0)$ as well as non-Newtonian $(\alpha=-0.05,-0.1)$ cases. It is evident from the Figures 2 and 3 that both R_0 and R_1 increase with the increase of Hartmann number (M) for both Newtonian and non-Newtonian cases. But these results decrease with the increase with the increase of Hall current (m).

The amplitude and the phase difference of the shear stress at the stationary plate $(\eta=0)$ for the steady flow can be obtained as,

$$\tau_{0r}=\sqrt{\tau_{0x}^2+\tau_{0y}^2}\ ,\quad \theta_{0r}=\tan^{-1}\left(\frac{\tau_{0y}}{\tau_{0x}}\right)$$

where

$$\tau_{0x}+i\,\tau_{0y}=\left(\frac{\partial q}{\partial\eta}\right)_{\eta=0}, \tag{28}$$

$$\tau_{0x}=\left[\frac{\partial u_0}{\partial\eta}+\alpha\left(\omega\frac{\partial^2u_0}{\partial\eta\partial t}+s\frac{\partial^2u_0}{\partial\eta^2}\right)\right]_{\eta=0},$$

$$\tau_{0y}=\left[\frac{\partial \mathrm{v}_0}{\partial\eta}+\alpha\left(\omega\frac{\partial^2\mathrm{v}_0}{\partial\eta\partial t}+s\frac{\partial^2\mathrm{v}_0}{\partial\eta^2}\right)\right]_{\eta=0} \tag{29}$$

Here τ_{0x} and τ_{0y} are, respectively, the shear stresses at the stationary plate due to the primary and the secondary velocity components.

Table 1: Values of τ_{0r} and θ_{0r} for various values of $m,\ M$ with $\Omega=8,\ s=0.3,\ k=0.2$.

Case	m	M	α	τ_{0r}	θ_{0r}
I	1	2	0	2.7482	0.7864
			-0.05	3.4562	0.6842
			-0.10	4.2684	0.5678
II	2	1	0	2.5684	0.8997
			-0.05	3.2772	0.8224
			-0.10	4.1282	0.7826
III	4	1	0	4.1226	0.6422
			-0.05	6.2254	0.5864
			-0.10	8.7265	0.5216

Table 1 exhibits the effects of the visco-elastic parameter α on the amplitude τ_{0r} and the phase difference θ_{0r} of the shear stress at the stationary plate $(\eta=0)$ for the steady part of the flow with the combination of the other flow parameters M and m. It is observed from the table 1 that the values of τ_{0r} increase but θ_{0r} decrease with the increasing values of the non-Newtonian parameter $(\alpha = 0, -0.05, -0.1)$. This table shows that τ_{0r} increase with the increase of Hartmann number (M) but decrease with the increase of Hall current (m) for both Newtonian as well as non-Newtonian cases.

The solutions of q_1 and q_2 together give the unsteady part of the flow. The unsteady primary and secondary velocity components $u_1(\eta)$ and $v_1(\eta)$, respectively, for the fluctuating flow can be obtained as

$$u_1(\eta,t) = \{\text{Real}\, q_1(\eta) + \text{Real}\, q_2(\eta)\}\cos t - \{\text{Im}\, q_1(\eta) - \text{Im}\, q_2(\eta)\}\sin t \tag{30}$$

$$v_1(\eta,t) = \{\text{Real}\, q_1(\eta) - \text{Real}\, q_2(\eta)\}\sin t + \{\text{Im}\, q_1(\eta) + \text{Im}\, q_2(\eta)\}\cos t \tag{31}$$

The resultant velocity or amplitude and the phase difference of the unsteady flow are given by

$$R_1 = \sqrt{u_1^2 + v_1^2}\ , \ \theta_1 = \tan^{-1}\left(\frac{v_1}{u_1}\right) \tag{32}$$

For the unsteady part of the flow, the amplitude and phase difference of shear stresses at the stationary plate $(\eta = 0)$ can be obtained as

$$\tau_{1x} + \tau_{1y} = \left(\frac{\partial u_1}{\partial \eta}\right)_{\eta=0} + i\left(\frac{\partial v_1}{\partial \eta}\right)_{\eta=0} \tag{33}$$

which gives

$$\tau_{1r} = \sqrt{\tau_{1x}^2 + \tau_{1y}^2} \quad , \theta_{1r} = \tan^{-1}\left(\frac{\tau_{1y}}{\tau_{1x}}\right) \tag{34}$$

Table 2: Values of τ_{1r} and θ_{1r} for various values of m, M with $\Omega = 8,\ s = 0.3,\ k = 0.2$.

Case	m	M	α	τ_{1r}	θ_{1r}
I	1	2	0	1.6412	0.5262
			-0.05	3.8926	0.4721
			-0.10	6.2542	0.3946
II	2	1	0	1.5864	0.5987
			-0.05	3.1225	0.5678
			-0.10	5.8462	0.4996
III	4	1	0	2.5778	0.4884
			-0.05	7.8892	0.4263
			-0.10	10.7884	0.3692

The amplitude τ_{1r} and phase difference θ_{1r} of the unsteady shear stresses at the stationary plate $(\eta = 0)$ have been listed in Table 2. It is observed from the Table 2 that the values of τ_{1r} decrease with the increase of Hall current (m). But τ_{1r} increase with the increase of Hartmann number (M) for both Newtonian as well as non-Newtonian cases. The phase difference θ_{1r} increase with the increase of m, but decrease with the increase of M for both Newtonian and non-Newtonian cases.

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