A Subclass of p −Valent and Analytic Functions Associated With Dzoik Srivastava Linear Operator

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Research Article

Abstract: In this paper, a new subclasses $\mathrm{H}(\mathrm{a},\mathrm{b},\mathrm{p},\sigma)$and $\mathrm{K}(\mathrm{a},\mathrm{b},\mathrm{p},\sigma)$ of analytic and p −valent functions has been introduced. Necessary and sufficient conditions for these classes are discussed. Distortion properties and the result of modified Hadamard product are obtained for the subclasses $\mathrm{H}(\mathrm{a},\mathrm{b},\mathrm{p},\sigma)$and $\mathrm{K}(\mathrm{a},\mathrm{b},\mathrm{p},\sigma)$. Some other results for the same subclasses of functions are also obtained.

Key words: Hadamard product, Dzoik -Srivastava operator, generalized fractional operator, p −valent functions, analytic functions, linear operator, distortion theorem, Riemann-Lioueville operator.

1. Introduction

Let T denote the class of functions of the form

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n, (a_n \geq 0) \qquad (1.1)$$

which are analytic and p −valent in the unit disk $U = \{z : |z| < 1\}$.

Function $g(z) \in T$ is given by

$$g(z) = z^p + \sum_{n=k}^{\infty} b_n z^n , \quad (1.2)$$

then the Hadamard product (or Convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{n=k}^{\infty} a_n b_n z^n = (g * f)(z), \qquad z \in U. \quad (1.3)$$

Let A_p denote the subclass of T consisting of functions of the form

$$f(z) = z^p - \sum_{n=k}^{\infty} a_n z^n , (a_n \geq 0). \qquad (1.4)$$

The Generalized hypergeometric function ${}_qF_s(\cdot)$ for positive real values of $\alpha_1, \ldots\ldots, \alpha_q$ and $\beta_1, \ldots\ldots, \beta_s$ $(\beta_j \neq 0, -1, \ldots.; j = 1, 2, \ldots, s)$ is defined by

$${}_qF_s(\cdot) = {}_qF_s(\alpha_1, \ldots\ldots, \alpha_q;\ \beta_1, \ldots\ldots, \beta_s) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n, \ldots\ldots, (\alpha_q)_n\, z^n}{(\beta_1)_n, \ldots\ldots, (\beta_s)_n\, (n)!} \qquad (1.5)$$

$(q \leq s + 1; q, s \in N_0 = N\cup\{0\}; z \in U)$,

where $(\alpha)_n$ is the pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma a} = \begin{cases} 1, & (n = 0), \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in N). \end{cases} \qquad (1.6)$$

Corresponding to a function $h_p(\alpha_1, \ldots, \alpha_q ; \beta_1, \ldots , \beta_s; z)$ is defined by

$$h_p(\alpha_1, \ldots, \alpha_q ; \beta_1, \ldots , \beta_s; z) = z^p\, {}_qF_s(\alpha_1, \ldots, \alpha_q;\ \beta_1, \ldots, \beta_s; z), \quad (1.7)$$

The Dziok and Srivastava operator [6] $H_p(\alpha_1, \ldots, \alpha_q ; \beta_1, \ldots , \beta_s)$ is defined by

$$H_p(\alpha_1, \ldots, \alpha_q ; \beta_1, \ldots , \beta_s) f(z) = h_p(\alpha_1, \ldots, \alpha_q ; \beta_1, \ldots , \beta_s; z) * f(z)$$

$$= z^p - \sum_{n=k}^{\infty} H(n) a_n z^n \qquad (1.8)$$

where $H(n) = \dfrac{(\alpha_1)_{n-p} \cdots (\alpha_q)_{n-p}}{(\beta_1)_{n-p} \cdots (\beta_s)_{n-p}\, (n-p)!}$. $\qquad (1.9)$

If we set $p = s = 1$ and $q = 2$, then the linear operator $H_p(\alpha_1, \ldots, \alpha_q ; \beta_1, \ldots , \beta_s) f(z)$ reduces to linear operator $\mathfrak{F}(\alpha_1, \alpha_2, \beta_1) f(z)$ $(cf.[12])$ as

$$H(\alpha_1,\alpha_2,\beta_1)f(z) = z - \sum_{n=k}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}}{(\beta_1)_{n-1}\,(n-1)!} a_n z^n = \mathfrak{F}(\alpha_1,\alpha_2,\beta_1)f(z). \quad (1.10)$$

If we put $\alpha_2 = 1$ in above equation (1.10), then it reduces to Carlson-Shaffer operator [2] as

$$H(\alpha_1,1,\beta_1)f(z) = z - \sum_{n=k}^{\infty} \frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}} a_n z^n = \mathcal{L}(\alpha_1,\beta_1)f(z). \quad (1.11)$$

In particular, if we put $\alpha_1 = 1+\lambda$, $\beta_1 = 1$, then it reduces to Ruscheweyh operator [10] given by

$$H(1+\lambda,1;1)f(z) = D^{\lambda}f(z) = z - \sum_{n=k}^{\infty} \frac{(1+\lambda)_{n-1}}{(1)_{n-1}} a_n z^n. \quad (1.12)$$

If we set $\alpha_1 = 1+\lambda$, $\alpha_1 = 1$, $\beta_1 = v+2$, then (1.10) reduces to Bernadi-Libera-Livingston integral operator given earlier by (see [11, 1, 8]) as

$$H(1+\lambda,1,v+2)f(z) = J_v f(z) = z - \sum_{n=k}^{\infty} \frac{(1+\lambda)_{n-1}}{(v+2)_{n-1}} a_n z^n. \quad (1.13)$$

Now, we recall the following definition of fractional derivative operator due to Owa [9].

Definition1. The fractional integral of order λ, for function $f(z)$ is defined by

$$D_z^{-\lambda}f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt\,;\ \lambda > 0\,, \quad (1.14)$$

where the analytic function $f(z)$ is defined in a simply-connected region of the z -plane containing the origin, and multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

The fractional derivative operator of order λ, for an analytic function $f(z)$ is defined by

$$D_z^{\lambda}f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt\,;\ 0 \le \lambda < 1, \quad (1.15)$$

where the conditions, under which (1.15) is valid, are similar to those stated with (1.14).

Definition 2. Under the hypotheses of (1.15), the fractional derivative of function $f(z)$, order $n+\lambda$ is defined by

$$D_z^{n+\lambda}f(z) = \frac{d^n}{dz^n} D_z^{\lambda}f(z), (0 \le \lambda < 1; n \in N_0). \quad (1.16)$$

In particular case, if we let $\alpha_1 = 2, \alpha_2 = 1, \beta_1 = 2-\lambda$ then (1.10) reduces to linear operator $\Omega^{\lambda} : \mathbb{Q} \to \mathbb{Q}$ due to Srivastava and Owa [3] is defined by

$$\Omega^{\lambda}f(z) = H(2,1,2-\lambda)f(z) = z - \sum_{n=k}^{\infty} \frac{(2)_{n-1}}{(2-\lambda)_{n-1}} a_n z^n, \quad (1.17)$$

where $\Omega^{\lambda}f(z) = \Gamma(2-\lambda)z^{\lambda}\ D_z^{\lambda}f(z)$.

For details, one can see [4, 5].

Definition 3. A function $f(z) \in A_p$ defined by (1.4) is said to be in the class $H(a,b,p,\sigma)$ if it satisfied the following relations (cf. [7])

$$\left| \frac{\left(H_p(\alpha_1,\dots,\alpha_q\,;\beta_1,\dots,\beta_s)f(z)\right) - 1}{b\left(H_p(\alpha_1,\dots,\alpha_q\,;\beta_1,\dots,\beta_s)f(z)\right) - a} \right| < \sigma, (z \in \mathrm{U}) \quad (1.18)$$

$(-1 \le a < b \le 1; 0 < b \le 1; 0 < \sigma \le 1)$.

A function $f(z) \in A_p$ defined by (1.4) is said to be in the class $K(a,b,p,\sigma)$ if and only if

$$\frac{zf'(z)}{p} \in H(a,b,p,\sigma). \quad (1.19)$$

2. Coefficient Estimates

Theorem 2.1. Let the function $f(z)$ be defined by (1.4). Then $f(z) \in H(a,b,p,\sigma)$ if and only if

$$\sum_{n=k}^{\infty} H(n)(1+b\sigma)\ a_n \le (b-a)\sigma. \quad (2.1)$$

The result is sharp for the function

$$f(z) = z^p - \frac{(b-a)\sigma}{H(n)(1+b\sigma)} z^n\ , n \ge k \quad (2.2)$$

where $H(n)$ is defined by (1.9).

Proof. Let $f(z) \in H(a,b,p,\sigma)$. Then in view of (1.18), we have

$$\left|\frac{\left(H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z)\right)-1}{b\left(H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z)\right)-a}\right|<\sigma$$

$$\left|\frac{\left(z^p-\sum_{n=k}^{\infty} H(n)a_n z^n\right)-1}{b\left(z^p-\sum_{n=k}^{\infty} H(n)a_n z^n\right)-a}\right|<\sigma, \qquad (2.3)$$

by using $|\mathrm{Re}(z)| \le |z|$ in (2.3), we get

$$\mathrm{Re}\left\{\frac{z^p-\sum_{n=k}^{\infty} H(n)a_n z^n-1}{bz^p-b\sum_{n=k}^{\infty} H(n)a_n z^n-a}\right\}<\sigma,$$

taking values of z on the real axis and let $z \to 1^-$ through real values then

$$\sum_{n=k}^{\infty} H(n)a_n \le \sigma\left(b-b\sum_{n=k}^{\infty} H(n)a_n-a\right),$$

$$\sum_{n=k}^{\infty} H(n)(1+b\sigma)\ a_n \le (b-a)\sigma.$$

Conversely, let inequality (2.3) hold true, then

$$\left|\frac{\left(H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z)\right)-1}{b\left(H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z)\right)-a}\right|<\sigma$$

$$\left|\left(H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z)\right)-1\right|-\sigma\left|b\left(H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z)\right)-a\right|$$

$$\left|z^p-\sum_{n=k}^{\infty} H(n)\,a_n z^n-1\right|-\sigma\left|bz^p-b\sum_{n=k}^{\infty} H(n)\,a_n z^n-a\right|<\sum_{n=k}^{\infty} H(n)(1+b\sigma)\ a_n-(b-a)\sigma \le 0.$$

By maximum modulus principle, this implies that $f(z) \in H(a,b,p,\sigma)$.

$$\sum_{n=k}^{\infty} a_n=\frac{(b-a)\sigma}{H(n)(1+b\sigma)}, n \ge k$$

The result is sharp for the functions

$$f(z)=z^p-\frac{(b-a)\sigma}{H(n)(1+b\sigma)}z^n, \qquad n \ge k.$$

where $H(n)$ is defined by (1.9).

Theorem 2.2. Let the function $f(z)$ be defined by (1.4). Then $f(z) \in K(a,b,p,\sigma)$ if and only if

$$\sum_{n=k}^{\infty} n\,H(n)(1+b\sigma)\ a_n \le (b-a)\sigma\,. \qquad (2.4)$$

The result is sharp for the function

$$f(z)=z^p-\frac{(b-a)\sigma}{n\,H(n)(1+b\sigma)}z^n\,, \qquad n \ge k \qquad (2.5)$$

where $H(n)$ is defined by (1.9).

Proof. On using (1.18) and (1.19), we easily arrive at the desired result (2.4) and (2.5).

3. Closure theorem

Let the function $f_i(z)$ be defined for $i = 1, \dots, m$ by

$$f_i(z) = z^p - \sum_{n=k}^{\infty} a_{i,n} z^n, \left(a_{i,n} \geq 0;\ p \in N\right). \quad (3.1)$$

Theorem 3.1. Let the function $f_i(z)$ defined by (3.1) be in the class $H(a_i, b_i, p, \sigma)$, for each $i = 1, \dots, m$. Then the function $h(z)$ defined by

$$h(z) = z^p - \frac{1}{m} \sum_{n=k}^{\infty} \left(\sum_{i=1}^{m} a_{i,p} \right) z^n, \left(a_{i,n} \geq 0;\ p \in N\right) \quad (3.2)$$

is in the class $H(a, b, p, \sigma)$, where $a = \min_{1 \leq i < m} (a_i)$ and $b = \max_{1 \leq j < m} (b_j)$ (3.3)

Proof. Since $f_i(z) \in H(a_i, b_i, p, \sigma)$, then by using (2.1)

$$\sum_{n=k}^{\infty} H(n)(1 + b\sigma)\ a_{i,p} \leq (b - a)\sigma, \quad (3.4)$$

where $H(n)$ is given by (1.9). Therefore

$$\sum_{n=k}^{\infty} H(n) \left(\frac{1}{m} \sum_{i=1}^{m} a_{i,p} \right) (1 + b\sigma) = \sum_{n=k}^{\infty} H(n)(1 + b\sigma) \left(\frac{1}{m} \sum_{i=1}^{m} a_{i,p} \right)$$

$\leq (b - a)\sigma$ [by using (3.3)]

which shows that $h(z) \in H(a, b, p, \sigma)$.

The theorem is completely proved.

Theorem 3.2. Let the function $f_i(z)$ defined by (3.1) be in the class $K(a_i, b_i, p, \sigma)$ for each $i = 1, \dots, m$. Then the function $h(z)$ defined by

$$h(z) = z^p - \frac{1}{m} \sum_{n=k}^{\infty} \left(\sum_{i=1}^{m} a_{i,p} \right) z^n, \left(a_{i,n} \geq 0;\ p \in N\right). \quad (3.5)$$

is in the class $K(a, b, p, \sigma)$, where $a = \min_{1 \leq i < m} (a_i)$ and $b = \max_{1 \leq j < m} (b_j)$ (3.6)

Proof. The proof follows exactly on the same lines as that of Theorem 3.1.

4. Distortion theorem for the classes $H(a, b, p, \sigma)$ and $K(a, b, p, \sigma)$

Theorem 4.1. Let the function $f(z)$ defined by (1.4) be in the class $H(a, b, p, \sigma)$. Then

$$|f(z)| \leq |z|^p - \frac{(b - a)\sigma}{H(k)(1 + b\sigma)} |z|^k, \quad (4.1)$$

and

$$|f(z)| \geq |z|^p + \frac{(b - a)\sigma}{H(k)(1 + b\sigma)} |z|^k, \quad (4.2)$$

for $z \in U$, provided that $-1 \leq a < b \leq 1; 0 < b \leq 1; 0 < \sigma \leq 1$, where $H(k)$ is defined by (1.9).

Proof. by using(1.4), then we have

$$|f(z)| \geq |z|^p - \sum_{n=k}^{\infty} a_n |z|^n$$

$$|f(z)| \geq |z|^p - |z|^k \sum_{n=k}^{\infty} a_n$$

by using (2.1), we get

$$|f(z)| \geq |z|^p - \frac{(b - a)\sigma}{H(k)(1 + b\sigma)} |z|^k,$$

here $H(n)$ is defined by (1.9).

We know that $H(n)$ is non decreasing for $n \geq k$, then we have

$$0 < H(n) \leq H(k) = \frac{(\alpha_1)_{k-p} \cdots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}\ (k-p)!} = \frac{\prod_{j=1}^{q} (\alpha_q)_{k-p}}{\prod_{j=1}^{s} (\beta_s)_{k-p} (k-p)!} \quad (4.3)$$

Then,

$$|f(z)| \leq |z|^p - \frac{(b - a)\sigma\ (\beta_1)_{k-p} \dots (\beta_s)_{k-p}\ (k - p)!}{(1 + b\sigma)(\alpha_1)_{k-p} \dots (\alpha_q)_{k-p}} |z|^k, \quad (4.4)$$

and

$$|f(z)| \geq |z|^p + \frac{(b-a)\sigma(\beta_1)_{k-p} \dots (\beta_s)_{k-p}\,(k-p)!}{(1+b\sigma)(\alpha_1)_{k-p} \dots (\alpha_q)_{k-p}} |z|^k. \qquad (4.5)$$

The theorem is completely proved.

Corollary 4.2. Under the hypothesis of theorem (4.1), $f(z)$ is included in a disc with it's centre at the origin and radius r given by

$$r = 1 + \frac{(b-a)\sigma(\beta_1)_{k-p} \dots (\beta_s)_{k-p}\,(k-p)!}{(1+b\sigma)(\alpha_1)_{k-p} \dots (\alpha_q)_{k-p}}. \qquad (4.6)$$

Theorem 4.3. Let the function $f(z)$ defined by (1.4) be in the class $K(a, b, p, \sigma)$. Then

$$|f(z)| \leq |z|^p - \frac{(b-a)\sigma}{k\,H(k)(1+b\sigma)} |z|^k, \qquad (4.7)$$

and

$$|f(z)| \geq |z|^p + \frac{(b-a)\sigma}{k\,H(k)(1+b\sigma)} |z|^k, \qquad (4.8)$$

for $z \in U$, provided that $-1 \leq a < b \leq 1; 0 < b \leq 1; 0 < \sigma \leq 1$.
Where $H(k)$ is defined by (1.9).

Proof. On using (1.4) and (2.4), we easily arrive at the desired result (4.7) and (4.8).

Corollary 4.4. Under the hypothesis of theorem (4.2), $f(z)$ is included in a disc with it's centre at the origin and radius r' given by

$$r' = 1 + \frac{(b-a)\sigma(\beta_1)_{k-p} \dots (\beta_s)_{k-p}\,(k-p)!}{k(1+b\sigma)(\alpha_1)_{k-p} \dots (\alpha_q)_{k-p}}. \qquad (4.9)$$

Theorem 4.5. Let the function $f(z)$ defined by (1.4) be in the class $H(a, b, p, \sigma)$. Then

$$\left|H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) f(z)\right| \geq |z|^p - \frac{(b-a)\sigma}{(1+b\sigma)} |z|^k \qquad (4.10)$$

and

$$\left|H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) f(z)\right| \leq |z|^p + \frac{(b-a)\sigma}{(1+b\sigma)} |z|^k. \qquad (4.11)$$

Proof. By using (1.8), we get

$$\left|H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) f(z)\right| \geq |z|^p - \sum_{n=k}^{\infty} H(n) a_n |z|^n \geq |z|^p - |z|^k \sum_{n=k}^{\infty} H(n) a_n, \qquad (4.12)$$

where $H(k) = \frac{(\alpha_1)_{k-p} \dots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \dots (\beta_s)_{k-p}\,(k-p)!}$.

by using (2.1), (4.12) reduces to

$$\left|H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) f(z)\right| \geq |z|^p - \frac{(b-a)\sigma}{(1+b\sigma)} |z|^k$$

and

$$\left|H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) f(z)\right| \leq |z|^p + \frac{(b-a)\sigma}{(1+b\sigma)} |z|^k.$$

This completes the proof of theorem 4.5.

Theorem 4.6. Let the function $f(z)$ defined by (1.4) be in the class $K(a, b, p, \sigma)$. Then

$$\left|H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) f(z)\right| \geq |z|^p - \frac{(b-a)\sigma}{k\,(1+b\sigma)} |z|^k \qquad (4.13)$$

and

$$\left|H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) f(z)\right| \leq |z|^p + \frac{(b-a)\sigma}{k\,(1+b\sigma)} |z|^k. \qquad (4.14)$$

Proof. The proof follows exactly on the same lines as that of Theorem 5.1.

Corollary 4.7. Let the function $f(z)$ defined by (1.4) be in the class $H(a, b, p, \sigma)$ and let $\alpha_1 = p+1, \alpha_2 = 1, \beta_1 = p+1-\lambda$ in (4.10) and (4.11) then it reduces to (4.15) and (4.16) respectively given by

$$\left|D_z^\lambda f(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)} |z|^{-\lambda} \left\{ |z|^p - \frac{(b-a)\sigma}{(1+b\sigma)} |z|^k \right\} \qquad (4.15)$$

and

$$\left|D_z^\lambda f(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)} |z|^{-\lambda} \left\{ |z|^p + \frac{(b-a)\sigma}{(1+b\sigma)} |z|^k \right\}. \qquad (4.16)$$

Corollary 4.8. Let the function $f(z)$ defined by (1.4) be in the class $H(a, b, p, \sigma)$ and let $\alpha_1 = p+1, \alpha_2 = 1, \beta_1 = p+1+\lambda$ in (4.10) and (4.11) then it reduces to (4.17) and (4.18) respectively given by

$$\left| D_z^{-\lambda} f(z) \right| \geq \frac{\Gamma(p+1)|z|^{\lambda}}{\Gamma(p+1+\lambda)} \left\{ |z|^p - \frac{(b-a)\sigma}{(1+b\sigma)} |z|^k \right\} \qquad (4.17)$$

and

$$\left| D_z^{-\lambda} f(z) \right| \leq \frac{\Gamma(p+1)|z|^{\lambda}}{\Gamma(p+1+\lambda)} \left\{ |z|^p + \frac{(b-a)\sigma}{(1+b\sigma)} |z|^k \right\}. \qquad (4.18)$$

Corollary 4.9. Let the function $f(z)$ defined by (1.4) be in the class $K(a, b, p, \sigma)$ and let $\alpha_1 = p + 1, \alpha_2 = 1, \beta_1 = p + 1 - \lambda$ in (4.13) and (4.14) then it reduces to (4.19) and (4.20) respectively given by

$$\left| D_z^{\lambda} f(z) \right| \geq \frac{\Gamma(p+1)|z|^{-\lambda}}{\Gamma(p+1-\lambda)} \left\{ |z|^p - \frac{(b-a)\sigma}{k\,(1+b\sigma)} |z|^k \right\} \qquad (4.19)$$

and

$$\left| D_z^{\lambda} f(z) \right| \leq \frac{\Gamma(p+1)|z|^{-\lambda}}{\Gamma(p+1-\lambda)} \left\{ |z|^p + \frac{(b-a)\sigma}{k\,(1+b\sigma)} |z|^k \right\}. \qquad (4.20)$$

Corollary 4.10. Let the function $f(z)$ defined by (1.4) be in the class $K(a, b, p, \sigma)$ and let $\alpha_1 = p + 1, \alpha_2 = 1, \beta_1 = p + 1 + \lambda$ in (4.13) and (4.14) then it reduces to (4.21) and (4.22) respectively given by

$$\left| D_z^{-\lambda} f(z) \right| \geq \frac{\Gamma(p+1)|z|^{\lambda}}{\Gamma(p+1+\lambda)} \left\{ |z|^p - \frac{(b-a)\sigma}{k\,(1+b\sigma)} |z|^k \right\} \qquad (4.21)$$

and

$$\left| D_z^{-\lambda} f(z) \right| \leq \frac{\Gamma(p+1)|z|^{\lambda}}{\Gamma(p+1+\lambda)} \left\{ |z|^p + \frac{(b-a)\sigma}{k\,(1+b\sigma)} |z|^k \right\}. \qquad (4.22)$$

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