A Bivariate Optimal Replacement Policy for a Repairable System Using Two Monotone Processes

A. Mallikarjuna Reddy[1], B. Venkata Ramudu[2*], M. Bhagya Lakshmi[3]

{[1]Professor, [3]Research Scholar} Department of Mathematics, S. K. University, Anantapur-515 003, Andhra Pradesh, INDIA.

[2]Assistant Professor, Department of Statistics, S.S.B.N. Degree College (Autonomous), Anantapur-515 001, Andhra Pradesh, INDIA.

[*]Corresponding Address:

venkataramudussbn@gmail.com

Research Article

Abstract: This paper studies a bivariate repairable system with one repairman is studied. Assume that the system after repair is not 'as good as new' and also the successive working times form a decreasing α-series process, the successive repair time's form an increasing geometric process and both the processes are exposing to Weibull failure law. Under these assumptions, we study an optimal replacement policy N in which we replace the system when the number of failures of the system reaches N. We derive an explicit expression for the long-run average cost per unit of time for the bivariate replacement policy (T,N) under which we replace the system when the number of failures reaches N or the age of the system reaches T or whichever occurs first. Under some mild conditions, we determine an optimal repair replacement policy N* such that the long run average cost per unit time is minimized. Numerical results are provided to support the theoretical results.

Key Words: Convolution, Geometric process, Monotone Processes- α-series process, Repair replacement policy, Renewal Process.

1. Introduction

Most maintenance models developed under the common assumption is that repair is perfect, i.e., repair restores the system to a "good as new" condition, and it is called a perfect repair model. However, perfect repair is not always satisfied. In practical application, most repairable systems are deteriorating due to ageing and accumulative wear. Barlow and Hunter [1] introduced a minimal repair model in which the repair activity doesn't change the failure rate of the system. Brown and Proschan [3] proposed an imperfect repair model in which the repair is equivalent to a perfect repair with probability 'p' and to a minimal repair with probability 1-p ($0 \le p \le 1$). Much research work has been carried out by Lam [7], Stadje and Zukerman [9], Stanley [10], Wang and Zhang [11, 12], Zhang et.al [15, 17] , Zhang [13,16], Zhang and Wang [18] and others along this direction. In application, because of the ageing effect and the accumulated wearing, most systems are degenerative in the sense that the consecutive **working time** between failures will be shorter and shorter, while the consecutive **repair time** after failures are getting longer and longer. In other words, the successive operating time are stochastically non- increasing, while the consecutive repair time are stochastically non-decreasing. To model such a deteriorating system, Lam [5,6] proposed a geometric process repair model. Using this model, Lam discussed two kinds of replacement polices known as **policy T** which based on the working age of the system while the other is called **policy N** which is based on the cumulative number of failures of the system. Under these two kinds of replacement polices explicit expressions are developed by minimizing the long run average cost per unit of time. Further he also showed that policy N is better than policy T. Zhang [14] also generalized Lam's [5,6] work using a bivariate replacement policy, called policy(T,N), under which the system is replaced at the working age T or at the time of the N^{th} failure, whichever occurs first, and proved that the bivariate policy is better than the uni-variate replacement policies, **policy N and policy T.** Later, Zhang [18] proposed optimal replacement policies for the maintenance problems by generalizing Zhang's [14] work.

Braun et.al [2] explained the increasing geometric process grows at most logarithmically in time, while the decreasing geometric process is almost certain to have a time of explosion. The α-series process grows either as a polynomial or exponential in time. It also noted that the geometric process doesn't satisfy a central limit theorem, while the α-series process does. Further, Braun et.al [4] presented that both the increasing geometric process and the α-series process have a finite first moment under certain general conditions. Thus the decreasing α-series process may be more appropriate for modeling system working times while the increasing geometric process is more suitable for modeling repair times of the system.

Based on this understanding, in this paper, we proposed two different monotone processes, which is a generalization to geometric processes proposed by Lam [5,6].Further, we discussed the model and optimal solution for a bivariate policy (T,N) under which the system is replaced at working age T or at the time of N^{th} failure, whichever occurs first exposing to two monotone process. The objective is to determine the optimal replacement policy (T,N)* such that the long run average cost per unit time is minimized. The explicit expression of the long run average cost per unit time is derived and the corresponding optimal replacement policy can be determined analytically or numerically. We prove that the optimal policy (T,N)* is better than the optimal policy N* for a simple repairable system.

In modeling of these deteriorating systems we utilize definitions given in Lam [5] .

Definition 1: Given two random variables X and Y, if P(X>t) > P(Y>t) for all real t, then X is called stochastically larger than Y or Y is stochastically less than X. This is denoted by $X >_{st} Y$ or $Y <_{st} X$ respectively.

Definition 2: Assume that $\{Y_n, n=1,2,\ldots\}$, is a sequence of independent non-negative random variables. If the distribution function of X_n is $F_n(t) = F(a^{n-1}t)$ for some a > 0 and all n=1,2,3,…., then $\{Y_n, n=1,2,\ldots\}$ is called a geometric process, '**a**' is the ratio of the geometric process.

Obviously:

if a>1, then $\{Y_n, n=1,2,\ldots\}$ is stochastically decreasing, i.e, $Y_n >_{st} Y_{n+1}$, n=1,2,…;

if 0<a<1, then $\{Y_n, n=1,2,\ldots\}$ is stochastically increasing, i.e, $Y_n <_{st} Y_{n+1}$, n=1,2,;

if a=1, then the geometric process becomes a renewal process.

Definition 3: Assume that $\{X_n, n=1,2,\ldots\}$, is a sequence of independent non-negative random variables. If the distribution function of X_n is $F_n(t) = F(k^{\alpha}t)$ for some $\alpha > 0$ and all n=1, 2, 3… then $\{X_n, n=1, 2\ldots\}$ is called an α series process, α is called exponent of the process. Braun *et. al* [12].

Obviously:

if $\alpha>0$, then $\{X_n, n=1,2,\ldots\}$ is stochastically decreasing, i.e, $X_n >_{st} X_{n+1}$, n=1,2,…;

if $\alpha<0$, then $\{X_n, n=1,2,\ldots\}$ is stochastically increasing, i.e., $X_n <_{st} X_{n+1}$, n=1,2,;

if $\alpha=0$, then the α series process becomes a renewal process.

2. The Model

In this section, we develop a model for the bi-variate optimal replacement policy for the system specializing to two monotone processes by minimizing the long-run expected reward per unit time with the following assumptions:

ASSUMPTIONS

1. At the beginning, a new system is installed. The system will be replaced some time by a new and identical one, and the replacement time is negligible.
2. Let X_n and Y_n be the successive working time follows decreasing a α-series process and the successive repair time's form an increasing geometric process respectively and both the processes are exposing to weibull failure law with parameters λ and μ respectively. A sequence $\{x_n, n=1,2,\ldots\}$ and sequence $\{Y_n, n=1, 2, \ldots\}$ are independent.
2. Let $F(K^{\alpha}x)$ and $G(a^{(n-1)}y)$ be the distribution function of Xn and Y_n respectively, and both are distributed according to weibull failure law.
3. $E(X_k) = \frac{\lambda}{k^{\alpha}}$ and $E(Y_k) = \frac{\mu}{a^{k-1}}$ where $\alpha > 0,\ 0 < a < 1.$
4. The constant repair cost rate is C_r and the constant revenue rate whenever the system working is C_w and the replacement cost is C.
5. The replacement policy (T, N) is used.

3. Optimal Solution

Using the assumptions on the model, it is determined an optimal replacement policy (T, N)* such that the long-run expected reward per unit time is maximum. Let T_1 be the first replacement time and let $T_n(n\geq2)$be the time between the $(n-1)^{th}$ replacement and n^{th} replacement. Clearly $\{T_1,T_2,\ldots\}$ forms a renewal process.

Let C(T,N) be the long-run average cost rate per unit time under the replacement policy (T,N). Since $\{T_1,T_2\ldots\}$ forms a renewal process, the inter arrival time between two consecutive replacements is a renewal cycle. Thus according to the renewal reward theorem [see Ross 8] we have:

$$C(N) = \frac{E(\text{cost incurred in a renewal cycle})}{\text{Length of cycle}}. \tag{3.1}$$

Let L be the length of a renewal cycle of the system under policy (T.N). Then

$$L = L_1 + L_2 \tag{3.2}$$

$$L_1 = T + \sum_{j=1}^{n} Y_j I\{U_n > T\} \tag{3.3}$$

$$L_2 = \left(\sum_{n=1}^{N} X_n + \sum_{n=1}^{N-1} Y_n\right) I(u_N \leq T) \tag{3.4}$$

I is the indicator function such that

I_A = 1, if event A occurs

= 0, if event a doesn't occurs.

Now the expected length of a renewal cycle L can be evaluated as follows:

$$E(L) = E\left[L_1 I_{\{U_N > T\}}\right] + E\left[L_2 I_{\{U_N \leq T\}}\right]. \tag{3.5}$$

Lemma: Let $U_N = \sum_{i=1}^{n} X_i^{(1)} = U_n + W_{N-n},\ n = 1,2,\ldots N-1$ then the expectation of indicator function $I\{U_n \leq T < U_N\}$ is given by:

$$E\left[I_{\{Un< T< UN\}}\right]=\int_0^T \bar{F}_{N-n}(a^n(T-t)dF_n(t)=F_n(T)-F_N(T), \text{ where n=1,2,....N-1.} \quad (3.6)$$

$$E(L_1)=E\left(T+\sum_{j=1}^{N}Y_j I\{U_n>T\}\right) \quad (3.7)$$

$$E(L_2)=E\left(\sum_{n=1}^{N}X_n+\sum_{n=1}^{N-1}Y_n\right)E\{I(u_N\le T)\} \quad (3.8)$$

Now, using equation (3.6) in the above lemma, we have:

$$E\left[L_1 I_{\{U_N>T\}}\right]=T\bar{F}_N(T)+\sum_{n=1}^{N-1}E\left(Y_n^{(1)}\right)\left[F_n(T)-F_N(T)\right], \quad (3.9)$$

$$E(L_2)=E\left(\sum_{n=1}^{N}X_n+\sum_{n=1}^{N-1}Y_n\right)F_N(T) \quad (3.10)$$

$$=\sum_{n=1}^{N}E(X_n)F_N(T)+\sum_{n=1}^{N-1}E(Y_n)F_N(T) \quad (3.11)$$

Substitute the results in equations (3.9) and (3.11), in equation (3.5)

$$E(L)=T\bar{F}_N(T)+\sum_{n=1}^{N-1}E(Y_n)F_n(T)+\sum_{n=1}^{N}E(X_n)F_N(T) \quad (3.12)$$

Thus from equation (3.1) and (3.12) the long-run expected average per unit time C(T,N) under policy (T,N) is given by:

$$C(T,N)=\frac{C_r\sum_{n=1}^{N-1}E(Y_n)F_n(T)+C_w\sum_{n=1}^{N}E(X_n)F_N(T)-C}{\sum_{n=1}^{N}E(X_n)F_N(T)+\sum_{n=1}^{N-1}E(Y_n)F_n(T)} \quad (3.13)$$

When T→∞, the optimal replacement policy (T,N*) reduces to C(N) and which is given by:

$$C(N,\infty)=\frac{C_r\sum_{n=1}^{N-1}E(Y_n)F_n(\infty)+C_w\sum_{n=1}^{N}E(X_n)F_N(\infty)-C}{\sum_{n=1}^{N}E(X_n)F_N(\infty)+\sum_{n=1}^{N-1}E(Y_n)F_n(\infty)}$$

$$C(N)=\frac{C_r\sum_{n=1}^{N-1}E(Y_n)+C_w\sum_{n=1}^{N}E(X_n)-C}{\sum_{n=1}^{N}E(X_n)+\sum_{n=1}^{N-1}E(Y_n)} \quad (3.14)$$

Now the expected length of working time can be obtained as follows:

Let

$$X_k^{(i)}\sim W\ (x:\eta_1,\beta_1),\ \ for\ k=1,2,3,....,\ \ and\ i=1,2.$$

Then the distribution function of $X_k^{(i)}$, for k=1,2,3,….and i=1,2 is :

$$F_k(x)=F(k^{\alpha}x)=1-e^{-\left(\frac{k^{\alpha}x}{\eta_1}\right)^{\beta^1}};\ x>0,\ \beta_1<1.$$

By definition the expected length of working time is given by :

$$E\left(X_x^{(i)}\right)=\int_0^{\infty}xdF\left(k^{\alpha}x\right),\qquad i=1,2. \quad (3.15)$$

$$=\frac{\eta_1\Gamma\left(1+\frac{1}{\beta_1}\right)}{k^{\alpha}}=\frac{\lambda}{k^{\alpha}},\ \ where\ \ \lambda=\eta_1\Gamma\left(1+\frac{1}{\beta_1}\right),\qquad i=1,2. \quad (3.16)$$

The expected length of repair time of component 1 can be obtained as follows:

Let $Y_k^{(i)}\sim W\ (y:\eta_2,\beta_2)$ then the distribution function of $Y_k^{(i)}$ for i=1,2, and k=1, 2, 3, …., is given as:

$$F_k(y)=F(a^{k-1}y)=1-e^{-\left(\frac{a^{k-1}y}{\eta_2}\right)^{\beta_2}};\ y>0,\beta_2<1.$$

By definition, the expected length of repair time is:

$$E\left(Y_x^{(i)}\right)=\int_0^{\infty}ydF\ (a^{k-1}y)\qquad ,i=1,2. \quad (3.17)$$

$$=\frac{\eta_2\Gamma\left(1+\frac{1}{\beta_2}\right)}{a^{k-1}}=\frac{\mu}{a^{k-1}},\quad where\ \mu=\eta_2\Gamma\left(1+\frac{1}{\beta_2}\right),\ i=1,2. \quad (3.18)$$

Using equations (3.15) and (3.18), we have:

$$C(N)=\frac{C_r\sum_{n=1}^{N-1}\frac{\mu}{b^{n-1}}-C_w\sum_{n=1}^{N}\frac{\lambda}{n^{\alpha}}-C}{\sum_{n=1}^{N-1}\frac{\mu}{b^{n-1}}+\sum_{n=1}^{N}\frac{\lambda}{n^{\alpha}}} \quad (3.19)$$

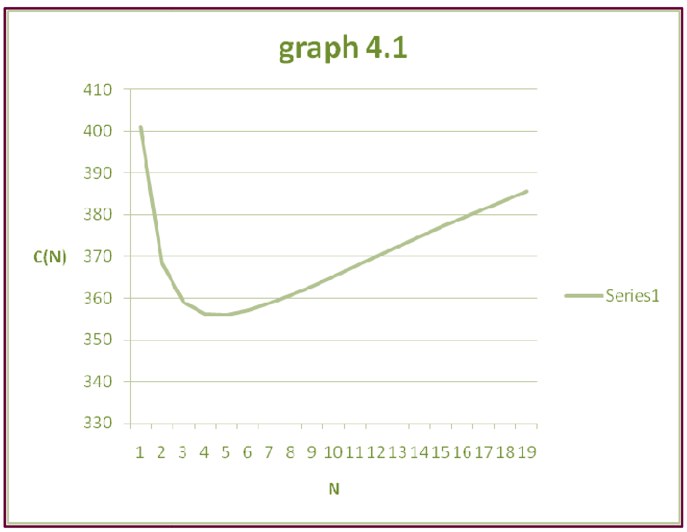
In the next section, using this C(N) equation given in (3.19), we determine the optimal value N* through empirical results.

4. Numerical Results and Conclusions

For the given hypothetical values of the parameters λ, μ,α , b, C_w, C_r, and C the long-run average cost per unit time is determined and is given in the following table: 4.1.

Table 4.1: Values of the long-run average cost per unit time

	For the given hypothetical values λ=10,μ=25,α=0.25, C_w=100,C_r=450, C=8000 b=0.95	For the given hypothetical values λ=10,μ=25,α=0.25, C_w=100,C_r=450, C=8000 b=0.90
N	C(N)	C(N)
2	401.0486	401.0486
3	368.4716	369.9845
4	359.0672	362.6706
5	356.2385	**362.0414**
6	**356.0372**	364.0167
7	357.0761	367.1586
8	358.7558	370.8445
9	360.7815	374.7673
10	362.9947	378.7621
11	365.3047	382.7345
12	367.6569	386.6284
13	370.0176	390.4094
14	372.365	394.0566
15	374.6852	397.5576
16	376.9691	400.905
17	379.2106	404.0958
18	381.4059	407.1289
19	383.5526	410.0056
20	385.6492	412.7282



Conclusions

1. From the table 4.1 and graph 4.1, it is examined that the long-run average cost per unit time C (6) = 356.0372 is minimum for the given b=0.95. Thus, we should replace the system at the time of 6^{th} failure.
2. Hor different values of the parameters, we observed that as 'b' increases number of failures increases, while 'α' decreases an increase in the number of failure, which coincides with the practical analogy and helps the decision maker for making an appropriate decision.

References

1. Barlow, R.E and Hunter, L.C., 1959, 'Optimum Preventive Maintenance Policies', Operations Research, 08, pp.90-100.
2. Braun, W.J., Li, W., and Zhao, Y.Q., 2005, 'Properties of geometric and related process', Naval Research Logistic, 52, pp.607-616.
3. Brown, M., and Proschan, F., 1983, 'Imperfect Repair', Journal of Applied Probability, 28, pp.851-859.
4. Braun, W.J., Li, W., and Zhao, Y.Q., 2008, 'Some theoretical Properties of the geometric and α-series process', Communication in Statistics- Theory and Methods, 37, pp.1483-1496.
5. Lam, Y., 1988a'A note on the optimal replacement problem', Advanced Applied Probability, 20, pp.479-482.

6. Lam, Y., 1988b 'Geometric Process and Replacement Problem', Acta, Mathematicae Applicatae Sinica, 4, pp.366-377.
7. Lam, Y., 2003, 'A geometric process maintenance model', South East Asian Bulletin of Mathematics, 27, pp.295-305 .
8. Ross, S.M., 1970, 'Applied Probability Models with Optimization Applications',San-Franciso, Holden-Day.
9. Stadje, W., and Zuckerman, D., 1992,'Optimal repair policies with general degree of repair in two maintenance models', Operations Research Letters, 11,pp.77-80.
10. Stanley, A.D.J., 1993,'On geometric process and repair replacement problems', Microelectronics and Reliability, Vol.33(4), pp.489-491.
11. Wang, G.J., and Zhang, Y.L., 2006, 'Optimal periodic preventive repair and replacement policy assuming geometric process repair', IEEE Transactions on Reliability, 55, pp.118-121.
12. Wang, G.J., and Zhang, Y.L., 2007, 'An optimal replacement policy for a two-component series system assuming geometric process repair', Computers and Mathematics with Applications, .54, pp.192-202.
13. Zhang, Y.L., 1999, 'An optimal geometric process model for a cold standby repairable system', Reliability Engineering and System Safety, 63, pp.107-110.
14. Zhang, Y.L., 1994, 'A bivariate optimal replacement policy for a repairable system', Journal of Applied probability, 31, pp. 1123-1127.
15. Zhang, Y.L., Yam, R.C.M., and Zuo, M.J., 2001, 'Optimal Replacement Policy for a deteriorating production system with preventive maintenance', International Journal of System Science,.32, pp.1193-1198.
16. Zang, Y.L., 2002, 'A geometric process repair model with good-as-new preventive repair',IEEE Transaction on Reliability, 51, pp. 223-228.
17. Zhang, Y.L., Wang, G.J., and Ji Z.C., 2006, 'Replacement problems for a cold standby repairable system', International Journal of System Science, 37, pp.17-25.
18. Zhang, Y.L., and Wang, G.J., 2006, 'A bivariate optimal repair-replacement model using geometric process for a cold standby repairable system', Engineering Optimization,.38, pp.609-619.