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Triangular Approximation of Fuzzy Numbers

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Research Article

Abstract: To manipulate with fuzzy numbers is not an easy task. Computational demands for dealing with them are greater. In this paper, we have introduced the new notation for a triangular fuzzy numbers. This notation computationally eases the algebraic operations of fuzzy numbers. A concept of epsilon-fuzzy number $r_\in$ on $\mathbb{R}$ which is a special type of triangular fuzzy number is introduced. The arithmetic operations on epsilon-fuzzy numbers are defined and some properties are obtained. Particularly, we have defined addition, subtraction and multiplication of triangular fuzzy numbers.

Keywords: Fuzzy set; Fuzzy number; Epsilon-fuzzy number.

1. Introduction

Fuzzy numbers were first introduced by Zadeh in 1975. There after theory of fuzzy numbers was further studied and developed by Dubois and Prade, R. Yager, Mizomoto, J. Buckly and many others. Fuzzy numbers plays an important role in many applications. But the main hurdle in the development of applications is the computational complexity. Particularly arithmetic operations on fuzzy numbers are not an easy task. Hence more attention is needed to simplify arithmetic computation with fuzzy numbers. By restricting fuzzy numbers to triangular fuzzy numbers addition and subtraction becomes simpler but still the operation of multiplication and division remains complex process. Therefore, some approximate methods are needed to simplify multiplication operation in particular. In this paper we have obtained one such approximation for multiplication which satisfies some criteria for approximation.

2. Preliminaries

Throughout this paper I stands for the interval [0, 1]. A fuzzy subset A of a set X is a function $A : X \to I$. If $\alpha \in I$, then the set $\{x \in X \mid A(x) \geq \alpha\}$ is called α-level cut or in short α-cut of A and is denoted by A_α. The strict α-level cut of A is the Support of A is the set $A_{0+} = \{x \in X \mid A(x) > 0\}$. If $A(x) = 1$, for some $x \in X$, then A is called normal fuzzy set. If each α-cut of A is convex then the fuzzy set A is called convex fuzzy set. We assume X = ℝ. A fuzzy number A is a subset of the real line ℝ with membership function $A : \mathbb{R} \to [0,1]$ such that A is normal, A is fuzzy convex and upper semi-continuous i.e. α-cut is closed for all α∈ [0, 1]. Support of A is bounded. If left hand curve and right hand curve are straight lines then the fuzzy number is called trapezoidal fuzzy number. The triangular fuzzy number is a particular type of a trapezoidal fuzzy number in which core is a singleton set. The membership function of a triangular fuzzy number is of the form

$$A(x) = \begin{cases} \dfrac{(x-l)}{(m-l)} & when\; l < x \leq m \\ \dfrac{u-x}{(u-m)} & when\; m < x \leq u \\ 0, & otherwise \end{cases}$$

We denote the above triangular fuzzy number by $A = (l, m, n)$.The fuzzy numbers are used to represent uncertain and incomplete information in decision making, linguistic controllers, expert systems etc. To obtain arithmetic computations with fuzzy numbers in more simpler way it is require to get nearest interval, triangular or trapezoidal approximation of fuzzy numbers which satisfies certain criteria. In this paper we obtain a triangular approximation of multiplication of two epsilon-fuzzy numbers which preserves value, ambiguity and width.

For a fuzzy number A every α-cut is a closed interval. Let $A\alpha = [AL\ (\alpha),\ AU\ (\alpha)]$, where, $A_L(\alpha) = \inf\{x \in R \mid A(x) > \alpha\}$ and $A_U(\alpha) = \sup\{x \in \mathbb{R} \mid A(x) > \alpha\}$.

The *value* of fuzzy number A is defined by $Val(A) = \int_0^1 \alpha(A_L(\alpha) + A_U(\alpha))d\alpha$.

The *ambiguity* of fuzzy number A is defined by $Amb(A) = \int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))\mathrm{d}\alpha$.

The *width* of fuzzy number A is defined by $w(A) = \int_0^1 (A_U(\alpha) - A_L(\alpha))\mathrm{d}\alpha$.

For arbitrary fuzzy numbers A and B with α-cuts [$AL\ (\alpha), AU\ (\alpha)$] and [$BL\ (\alpha), BU\ (\alpha)$] respectively, the *distance* between A and B is defined as

$$d(A,B) = \sqrt{\int_0^1 (|A_L(\alpha) - B_L(\alpha)|^2 + |A_U(\alpha) - B_U(\alpha)|^2)\mathrm{d}\alpha}\ .$$

3. ε-Fuzzy Number on $\mathbb{R}$

We introduce special types of symmetric, triangular fuzzy numbers called as ε-fuzzy numbers.

Definition3.1. If r is a real number then ε-fuzzy number r_ε, for some $\varepsilon \in \mathbb{R}, (\varepsilon > 0)$ is a symmetric fuzzy set whose support is the interval $(r - \varepsilon, r + \varepsilon)$. Thus, ε-fuzzy number is a fuzzy set denoted by r_ε, is a function $r_\varepsilon : \mathbb{R} \to I$ defined by

$$r_\varepsilon(x) = \begin{cases} \dfrac{x - (r - \varepsilon)}{\varepsilon} & \text{when } r - \varepsilon < x \le r \\ \dfrac{(r + \varepsilon) - x}{\varepsilon} & \text{when } r < x \le r + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Diagrammatically r_ε is as shown below

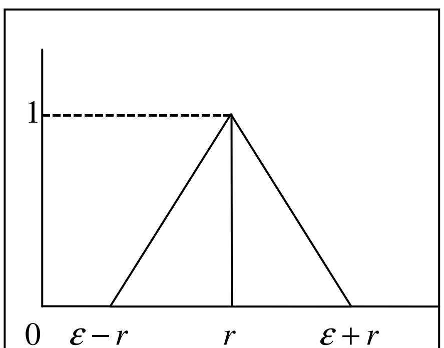

Fig. 1

The support of ε-fuzzy number r_ε is $(r - \varepsilon(1-\alpha), r + \varepsilon(1-\alpha)), r \in \mathbb{R}, \varepsilon \in \mathbb{R}$ and $\varepsilon > 0$. The α-cut of r_ε is denoted by $(r_\varepsilon)_\alpha = [r - \varepsilon(1-\alpha), r + \varepsilon(1-\alpha)]$, therefore $AL(\alpha) = r - \varepsilon(1-\alpha)$, $AU(\alpha) = r + \varepsilon(1-\alpha)$, width (or length) of $(r_\varepsilon)_\alpha$ is denoted by $l(r_\varepsilon)_\alpha$ and it is defined by $l(r_\varepsilon)_\alpha = 2\varepsilon(1-\alpha)$, which is independent of r.

Note that for ε = 0, the 0-fuzzy number *r0* is the characteristic function of $\{r\}$.

3.2 Arithmetic with ε-fuzzy numbers

3.2.1 Addition

Addition of ε-fuzzy numbers r_ε and s_ε is $(r+s)_{2\varepsilon}$ -fuzzy number defined by $(r_\varepsilon + s_\varepsilon) = (r+s)_{2\varepsilon}$.

Note that for $(r+s)_{2\varepsilon}$

i) α-cut is

$$(r_\varepsilon)_\alpha + (s_\varepsilon)_\alpha = [r-\varepsilon(1-\alpha), r+\varepsilon(1-\alpha)] + [s-\varepsilon(1-\alpha), s+\varepsilon(1-\alpha)]$$
$$= [(r+s)-2\varepsilon(1-\alpha), (r+s)+2\varepsilon(1-\alpha)]$$
$$= ((r+s)_{2\varepsilon})_\alpha.$$

ii) Support is $((r+s)-2\varepsilon, (r+s)+2\varepsilon)$

iii) Core is $(r+s)$

iv)Value is $(r+s)$

v) Ambiguity is $\frac{2\varepsilon}{3}$

vi)Width is 4ε

These values are similar to the addition obtained by extension principle or by Dubois- Prade computational formulas.

3.2.2 Negation

Negation of ε-fuzzy numbers r_ε is $(-r)_\varepsilon$.

3.2.3 Subtraction

Subtraction of ε-fuzzy numbers r_ε and s_ε is $(r-s)_{2\varepsilon}$-fuzzy number defined by

$(r_\varepsilon - s_\varepsilon) = (r-s)_{2\varepsilon}$.

Note that

$$(r_\varepsilon)_\alpha - (s_\varepsilon)_\alpha = [r-\varepsilon(1-\alpha), r+\varepsilon(1-\alpha)] - [s-\varepsilon(1-\alpha), s+\varepsilon(1-\alpha)]$$
$$= [(r-s)-2\varepsilon(1-\alpha), (r-s)+2\varepsilon(1-\alpha)]$$
$$= ((r-s)_{2\varepsilon})_\alpha.$$

3.2.4 Multiplication

Let r_ε and s_ε be two ε-fuzzy numbers where supports of r_ε and s_ε lies in ℝ+.

a) *Multiplication by extension principle*

$(r_\varepsilon)_\alpha (s_\varepsilon)_\alpha = ((rs)_{(r+s)\varepsilon})_\alpha = [(rs)-(r+s)\varepsilon(1-\alpha)+\varepsilon^2(1-\alpha)^2, (rs)+(r+s)\varepsilon(1-\alpha)+\varepsilon^2(1-\alpha)^2]$. Note that left and right functions are not linear and hence the multiplication obtained by extension principle is not triangular fuzzy number.

b) *Multiplication by Dubois-Prade method*

$$r_\varepsilon \bullet s_\varepsilon = (r \bullet s)_{(r+s)\varepsilon}.$$

Note that multiplication obtained by Dubois-Prade method is a triangular fuzzy number, whose α-cuts are given by

$$(r_\varepsilon)_\alpha (s_\varepsilon)_\alpha = ((rs)_{(r+s)\varepsilon,(r+s)\varepsilon})_\alpha = [(rs)-(r+s)\varepsilon(1-\alpha), (rs)+(r+s)\varepsilon(1-\alpha)].$$

c) *Multiplication by epsilon approximation method*

$$r_\varepsilon \bullet s_\varepsilon = (r \bullet s)_{(r+s)\varepsilon-\frac{\varepsilon^2}{2},\ (r+s)\varepsilon+\frac{\varepsilon^2}{2}}.$$

Note that multiplication obtained by epsilon approximation method is a triangular fuzzy number, whose α-cuts are given by

$(r_\varepsilon)_\alpha (s_\varepsilon)_\alpha = ((rs)_{(r+s)\varepsilon-\frac{\varepsilon^2}{2},\ (r+s)\varepsilon+\frac{\varepsilon^2}{2}})_\alpha = [(rs)-((r+s)\varepsilon-\frac{\varepsilon^2}{2})(1-\alpha), (rs)+((r+s)\varepsilon+\frac{\varepsilon^2}{2})(1-\alpha)]$ Multiplication of two ε-fuzzy numbers need not be ε-fuzzy number, since multiplication is not symmetric triangular fuzzy number.

Proposition 1. Multiplication of ε-fuzzy numbers by epsilon approximation method is commutative and associative

Proof. Let r_ε, s_ε and t_ε be the ε-fuzzy numbers whose support lies in ℝ+. Then

$$r_\varepsilon \bullet s_\varepsilon = (r \bullet s)_{(r+s)\varepsilon - \frac{\varepsilon^2}{2},\ (r+s)\varepsilon + \frac{\varepsilon^2}{2}}$$

Since the expression is symmetric in *r* and *s* and multiplication and addition of real numbers are commutative we get $r_\varepsilon \bullet s_\varepsilon = s_\varepsilon \bullet r_\varepsilon$

Similarly, consider

$$(r_\varepsilon \bullet s_\varepsilon) \bullet t_\varepsilon = ((r \bullet s)_{(r+s)\varepsilon - \frac{\varepsilon^2}{2},\ (r+s)\varepsilon + \frac{\varepsilon^2}{2}}) \bullet t_\varepsilon$$

$$= \left(r \bullet s \bullet t\right)_{(rs)\varepsilon + t((r+s)\varepsilon - \frac{\varepsilon^2}{2}) - \frac{((r+s)\varepsilon - \frac{\varepsilon^2}{2})\varepsilon}{2},\ (rs)\varepsilon + t((r+s)\varepsilon + \frac{\varepsilon^2}{2}) + \frac{((r+s)\varepsilon + \frac{\varepsilon^2}{2})\varepsilon}{2}}$$

$$= \left(r \bullet s \bullet t\right)_{rs\varepsilon + rt\varepsilon + st\varepsilon - \frac{(r+s+t)\varepsilon^2}{2} + \frac{\varepsilon^3}{4},\ rs\varepsilon + tr\varepsilon + st\varepsilon + \frac{(r+s+t)\varepsilon^2}{2} + \frac{\varepsilon^3}{4}}$$

Since the expression is symmetric in *r*, *s* and *t* we have

$r_\varepsilon \bullet (s_\varepsilon \bullet t_\varepsilon) = (r_\varepsilon \bullet s_\varepsilon) \bullet t_\varepsilon$

Proposition 2. Epsilon approximation multiplication of ε-fuzzy numbers preserves core, value and ambiguity

Proof. Let r_ε, s_ε be the ε-fuzzy numbers whose supports lies in ℝ+ . Then by extension principle method

$Core\,(r_\varepsilon \bullet s_\varepsilon) = rs$, $Value\ (r_\varepsilon \bullet s_\varepsilon) = rs + \frac{\varepsilon^2}{6}$, $Amb_{(r_\varepsilon \bullet s_\varepsilon)} = \frac{\varepsilon(r+s)}{3}$.

By epsilon approximation method we get the following values

$Core\,(r_\varepsilon \bullet s_\varepsilon) = rs$, $Value\ (r_\varepsilon \bullet s_\varepsilon) = rs + \frac{\varepsilon^2}{6}$, $Amb_{(r_\varepsilon \bullet s_\varepsilon)} = \frac{\varepsilon(r+s)}{3}$.

This shows that epsilon approximation of multiplication of fuzzy numbers is computationally simple and better approximation.

Note: By Dubois-Prade method we have the following values

$Core\,(r_\varepsilon \bullet s_\varepsilon) = rs$, $Value\ (r_\varepsilon \bullet s_\varepsilon) = rs$, $Amb_{(r_\varepsilon \bullet s_\varepsilon)} = \frac{\varepsilon(r+s)}{3}$.

Proposition 3. Epsilon approximation multiplication of ε-fuzzy numbers is distributive over addition.

4. Conclusion

In this paper symmetric triangular fuzzy numbers are renamed as ε-fuzzy numbers and epsilon approximation of multiplication is defined. Further it is pointed out that this approximation is computationally simple and satisfies the various parameters of approximation. Also in the process of approximation the properties of multiplication are preserved.

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