On Equitable Coloring of Star Graph Families

Sujitha N.[1], Pauline Mary Helen M.[2]

[1]M. Phil. Scholar, [2]Associate Professor, Nirmala College, Coimbatore-641018, Tamil Nadu, INDIA.

Corresponding Addresses:

[1]sujithasanthoshkumar@gmail.com, [2]helvic63@yahoo.co.in

Research Article

Abstract: In this paper, we discuss the equitable coloring of well known graph families of central graph, middle graph and total graph. We obtain the equitable chromatic number for the above said graph families.

Keywords: Central graph, middle graph, total graph, equitable coloring.

1. Introduction

For a given graph $G = (V, E)$ we do an operation on G, by subdividing each edge exactly once and joining all the non adjacent vertices of G. The graph obtained by this process is called the central graph[1,12] of G denoted by $C(G)$. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ The middle graph [10] of G, denoted by $M(G)$ is defined as follows .The vertex set of $M(G)$ is $V(G) \cup E(G)$ .Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [1,2, 6, 10] of G, denoted by $T(G)$ is defined as follows. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$, and $x; y$ are adjacent in G .(iii) x is in $V(G)$, y is in $E(G)$, and $x; y$ are incident in G. The notion of equitable coloring [9], was introduced by Meyer in 1973. If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, \ldots V_k$ such that each V_i is an independent set and the condition $\left\| V_i \right| - \left| V_j \right\| \le 1$ holds for every pair $i \ne j$, then G is said to be equitably k-colorable. The smallest integer k for which G is equitable k-colorable is known as the equitable chromatic number[4,5,6,7,8,10,11] of G and denoted by $\chi_{=}(G)$.

2. Equitable Coloring on Central Graph of Star Graph

Theorem2.1: *The equitable chromatic number of Central graph of star graph* $K_{1,n}$, $\chi_{=}[C(K_{1,n})] = n$

Proof. Let $V(K_{1,n}) = \{v, v_1, v_2, \ldots\ldots, v_n\}$ where vv_i = $e_i (1 \le i \le n)$ .By the definition of central graph $C(K_{1,n})$ has the vertex set $V(K_{1,n}) \cup \{u_i / 1 \le i \le n\}$ where u_i is the vertex subset of the edges e_i .The vertex subset $\{v, v_1, v_2, \ldots\ldots, v_n\}$ of $V(K_{1,n})$ induces a clique on n vertices.

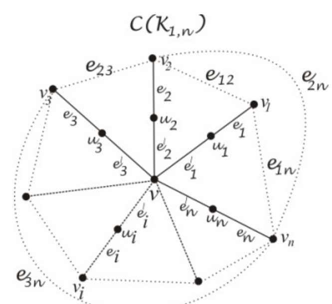

Figure 2(a): Star graph $K_{1,n}$

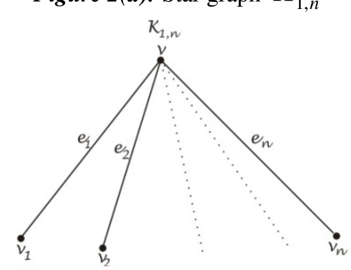

Figure 2(b): Central graph of star graph $K_{1,n}$

Let us prove the theorem by the method of mathematical induction. Let $P(n)$ denotes the equitable chromatic number of central graph of star graph $K_{1,n}$, $\chi_{=}[C(K_{1,n})] = n$, $\forall n \ge 3$ and $n \in N$.

Let $n = 3$, then $V(K_{1,3}) = \{v, v_1, v_2, v_3\}$, by the definition of central graph

$V[C(K_{1,3})] = \{v, v_1, v_2, v_3\} \cup \{u_1, u_2, u_3\}$. Here the vertex subset $\{v\}$ is adjacent to each $u_i (1 \le i \le 3)$ The vertex subset $\{v_1, v_2, v_3\}$ induces a clique on 3 vertices. By partitioning the vertex set $V[C(K_{1,3})]$, we get $V_1 = \{v, v_1\}, V_2 = \{u_1, v_2, u_3\}, V_3 = \{u_2, v_3\}$. Clearly $V_i's$ are disjoint and $\cup_{i=1}^{3} V_i = V[C(K_{1,3})]$ with $\left\| V_i \right| - \left| V_j \right\| \le 1$ holds for every pair $i \ne j$ .Therefore the equitable chromatic number of central graph of star graph $K_{1,3}$, $\chi_=[C(K_{1,3})] = 3$. Therefore $P(3)$ is true.

Let us assume that $P(k)$ is true .i.e., let us assume that the equitable chromatic number of central graph of star graph $K_{1,k}$, $\chi_=[C(K_{1,k})] = k \rightarrow (1)$ is true with the colour classes $V_1 = \{v, v_1\}, V_2 = \{u_1, v_2, u_k\}, V_i = \{u_{i-1}, v_i\}, 3 \le i \le k$. Clearly $V_i's$ are disjoint and $\cup_{i=1}^{k} V_i = V[C(K_{1,k})]$ with $\left\| V_i \right| - \left| V_j \right\| \le 1$ holds for every pair $i \ne j$

To prove $P(k+1)$ is true . i.e., to prove that the equitable chromatic number of central graph of star graph $K_{1,k+1}$, $\chi_=[C(K_{1,k+1})] = k+1$. The vertex set is given by

$$V[C(K_{1,k+1})] = \{v, v_1, v_2, ..., v_{k+1}\} \cup \{u_1, u_2, ..., u_{k+1}\}$$

$$= \{v, v_1, v_2, ..., v_k, u_1, u_2, ..., u_k\} \cup \{v_{k+1}, u_{k+1}\}$$

$$= V[C(K_{1,k})] \cup \{v_{k+1}, u_{k+1}\}$$

The vertex v_{k+1} and u_k requires a colour different from $V_i (1 \le i \le k)$ for proper colouring. Here the vertex v_{k+1} induces a clique with all $v_i (1 \le i \le k)$ and u_{k+1} is adjacent with v. The vertex u_{k+1} has the same colour of u_1 and v_2. By partitioning the vertex set of $C(K_{1,k+1})$, we get $V_1 = \{v, v_1\}, V_2 = \{u_1, v_2, u_{k+1}\}, V_i = \{u_{i-1}, v_i\}, 3 \le i \le k+1$. Clearly $V_i's$ are disjoint and $\cup_{i=1}^{k+1} V_i = V[C(K_{1,k+1})]$ with $\left\| V_i \right| - \left| V_j \right\| \le 1$ holds for every pair $i \ne j$ by using condition (1).It requires $(k)+1 = k+1$ colours. Therefore the equitable chromatic number of central graph of star graph $K_{1,k+1}$, $\chi_=[C(K_{1,k+1})] = k+1$. $P(k+1)$ is true whenever $P(k)$ is true. By the principle of mathematical induction $P(n)$ is true $\forall n \ge 3$ and $n \in N$ .Therefore the equitable chromatic number of central graph of star graph $K_{1,n}$, $\chi_=[C(K_{1,n})] = n$ $\forall n \ge 3$ and $n \in N$.

3. Equitable Coloring on Middle Graph of Star Graph

Theorem3.1: *The equitable chromatic number of middle graph of star graph* $K_{1,n}$, $\chi_=[M(K_{1,n})] = n+1$.

Proof. Let $V(K_{1,n}) = \{v, v_1, v_2, ..., v_n\}$ where $vv_i = e_i (1 \le i \le n)$ .By the definition of middle graph $M(K_{1,n})$ has the vertex set $V(K_{1,n}) \cup \{u_i / 1 \le i \le n\}$ where u_i is the vertex subset of the edges e_i .The vertex subset $\{u_i / 1 \le i \le n\}$ induces a clique among themselves. Hence v is adjacent to all $u_i's$ and each $u_i (1 \le i \le n)$ is adjacent to one another.

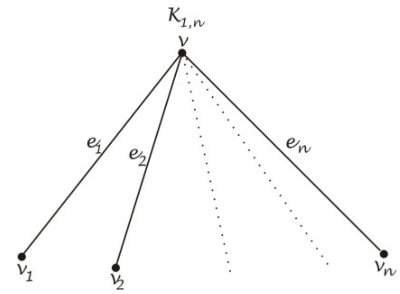

Figure 3(a): Star graph $K_{1,n}$

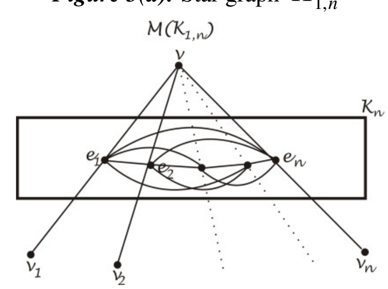

Figure 3(b): Middle graph of star graph $K_{1,n}$

Let us prove the theorem by the method of mathematical induction. Let $P(n)$ denotes the equitable chromatic number of middle graph of star graph $K_{1,n}$, $\chi_=[M(K_{1,n})] = n+1, \forall n \ge 3$ and $n \in N$

Let $n = 3$, then $V(K_{1,3}) = \{v, v_1, v_2, v_3\}$, by the definition of middle graph

$V[M(K_{1,3})] = \{v, v_1, v_2, v_3\} \cup \{u_1, u_2, u_3\}$. Here each vertex $u_i (1 \le i \le n)$ is adjacent with v and induces a clique among themselves. Hence we require four colours for proper colouring . By partitioning the vertex set $V[M(K_{1,3})]$, we get $V_1 = \{v, v_1\}, V_2 = \{u_1, v_2\}, V_3 = \{u_2, v_3\}, V_4 = \{u_3\}$. Clearly $V_i's$ are disjoint and $\cup_{i=1}^{4} V_i = V[M(K_{1,3})]$ with $\left\| V_i \right| - \left| V_j \right\| \le 1$ holds for every pair $i \ne j$. Therefore the equitable chromatic number of middle graph of star graph $K_{1,3}$, $\chi_{=}[M(K_{1,3})] = 4$

Therefore $P(3)$ is true. Let us assume that $P(k)$ is true .i.e., let us assume that the equitable chromatic number of middle graph of star graph $K_{1,k}$, $\chi_{=}[M(K_{1,k})] = k+1 \to (1)$ is true with the colour classes $V_1 = \{v, v_1\}, V_i = \{u_{i-1}, v_i\}, 2 \le i \le k, V_{k+1} = \{u_k\}$. Clearly $V_i's$ are disjoint and $\cup_{i=1}^{k+1} V_i = V[M(K_{1,k})]$ with $\left\| V_i \right| - \left| V_j \right\| \le 1$ holds for every pair $i \ne j$.

To prove P(k+1) is true . i.e., to prove that the equitable chromatic number of middle graph of star graph $K_{1,k+1}$, $\chi_{=}[M(K_{1,k+1})] = k+2$. The vertex set is given by

$$V[M(K_{1,k+1})] = \{v, v_1, v_2, ..., v_{k+1}\} \cup \{u_1, u_2, ..., u_{k+1}\}$$

$$= \{v, v_1, v_2, ..., v_k, u_1, u_2, ..., u_k\} \cup \{v_{k+1}, u_{k+1}\}$$

$$= V[M(K_{1,k})] \cup \{v_{k+1}, u_{k+1}\}$$

Here u_{k+1} induces a clique with all $u_i's$ and it is also adjacent with v, hence it requires a colour different from $V_i (1 \le i \le k+1)$ for proper colouring. The vertex v_{k+1} is non- adjacent with u_k and has the same colour as u_k. The colour class for $M(K_{1,k+1})$ is given by $V_1 = \{v, v_1\}, V_i = \{u_{i-1}, v_i\}, 2 \le i \le k+1, V_{k+2} = \{u_{k+1}\}$. Clearly $V_i's$ are disjoint and $\cup_{i=1}^{k+2} V_i = V[M(K_{1,k+1})]$ with $\left\| V_i \right| - \left| V_j \right\| \le 1$ holds for every pair $i \ne j$ by using condition (1).It requires $(k+1)+1 = k+2$ colours. Therefore the equitable chromatic number of middle graph of star graph $K_{1,k+1}$, $\chi_{=}[M(K_{1,k+1})] = k+2$. $P(k+1)$ is true whenever $P(k)$ is true. By the principle of mathematical induction $P(n)$ is true $\forall n \ge 3$ and $n \in N$ .Therefore the equitable chromatic number of middle graph of star graph $K_{1,n}$, $\chi_{=}[M(K_{1,n})] = n+1$ $\forall n \ge 3$ and $n \in N$

4. Equitable Coloring on Total Graph of Star Graph

Theorem 4.1: *The equitable chromatic number of total graph of star graph* $K_{1,n}$, $\chi_{=}[T(K_{1,n})] = n+1$.

Proof. Let $V(K_{1,n}) = \{v, v_1, v_2, ..., v_n\}$ where vv_i = $e_i (1 \le i \le n)$ .By the definition of total grap $T(K_{1,n})$ has the vertex set $V(K_{1,n}) \cup \{u_i / 1 \le i \le n\}$ where u_i is the vertex subset of the edges e_i.

The vertex subset $\{v\}$ induces a clique with the vertex subset $\{v, v_1, v_2, ..., v_n, u_1, u_2, ..., u_n\}$. Moreover each $u_i (1 \le i \le n)$ induces a clique among themselves.

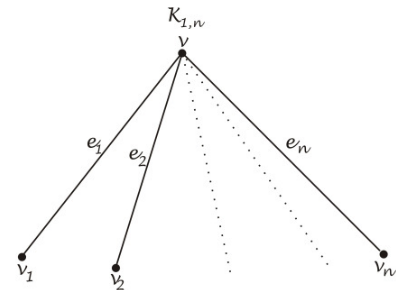

Figure 4(a): Star graph $K_{1,n}$

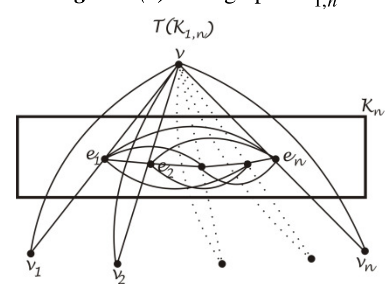

Figure 4(b): Total graph of star graph $K_{1,n}$

Let us prove the theorem by the method of mathematical induction. Let $P(n)$ denotes the equitable chromatic number of total graph of star graph $K_{1,n}$, $\chi_{=}[T(K_{1,n})] = n+1, \forall n \ge 3$ and $n \in N$

Let $n = 3$ then $V(K_{1,3}) = \{v, v_1, v_2, v_3\}$, by the definition of total graph

$V[T(K_{1,3})]=\{v,v_1,v_2,v_3\}\cup\{u_1,u_2,u_3\}$. Here the vertex subset $\{v\}$ is adjacent to all other vertices $\{v_1,v_2,v_3,u_1,u_2,u_3\}$ and each $u_i(1\le i\le 3)$ is adjacent to one another. Hence the vertex subset $\{v,u_1,u_2,u_3\}$ requires four different colours for proper colouring. By partitioning the vertex set $V[T(K_{1,3})]$, we get $V_1=\{v\},V_2=\{u_1,v_3\},V_3=\{u_2,v_1\},V_4=\{u_3,v_2\}$. Clearly $V_i's$ are disjoint and $\cup_{i=1}^{4} V_i = V[T(K_{1,3})]$ with $\left\|V_i\right|-\left|V_j\right\|\le 1$ holds for every pair $i\neq j$. Therefore the equitable chromatic number of total graph of star graph $K_{1,3}$, $\chi_=[T(K_{1,3})]=4$. Therefore $P(3)$ is true .

Let us assume that $P(k)$ is true .i.e., let us assume that the equitable chromatic number of total graph of star graph $K_{1,k}$, $\chi_=[T(K_{1,k})]=k+1\rightarrow(1)$ with the colour classes

$V_1=\{v\},V_2=\{u_1,v_k\},V_i=\{u_{i-1},v_{i-2}\},3\le i\le k+1$. Clearly $V_i's$ are disjoint and $\cup_{i=1}^{k+1} V_i = V[M(K_{1,k})]$ with $\left\|V_i\right|-\left|V_j\right\|\le 1$ holds for every pair $i\neq j$.

To prove $P(k+1)$ is true . i.e., to prove that the equitable chromatic number of total graph of star graph $K_{1,k+1}$, $\chi_=[T(K_{1,k+1})]=k+2$. The vertex set is given by

$V[T(K_{1,k+1})]=\{v,v_1,v_2,...,v_{k+1}\}\cup\{u_1,u_2,...,u_{k+1}\}$

$=\{v,v_1,v_2,...,v_k,u_1,u_2,...,u_k\}\cup\{v_{k+1},u_{k+1}\}$

$= V[T(K_{1,k})]\cup\{v_{k+1},u_{k+1}\}$

Here u_{k+1} induces a clique with all $u_i's$ and it is also adjacent with v, hence it requires a colour different from $V_i(1\le i\le k+1)$ for proper colouring. The vertex v_k and v_{k+1} has the same colour as u_{k+1} and u_1 respectively for equitable colouring. The colour class for $T(K_{1,k+1})$ is given by

$V_1=\{v\},V_2=\{u_1,v_{k+1}\},V_i=\{u_{i-1},v_{i-2}\},3\le i\le k+2.$ Clearly $V_i's$ are disjoint and $\cup_{i=1}^{k+2} V_i = V[T(K_{1,k+1})]$ with $\left\|V_i\right|-\left|V_j\right\|\le 1$ holds for every pair $i\neq j$ by using condition (1).

It requires $(k+1)+1=k+2$ colours. Therefore the equitable chromatic number of total graph of star graph $K_{1,k+1}$, $\chi_=[T(K_{1,k+1})]=k+2$. $P(k+1)$ is true whenever $P(k)$ is true. By the principle of mathematical induction $P(n)$ is true $\forall n\ge 3$ and $n\in N$ .Therefore the equitable chromatic number of total graph of star graph $K_{1,n}$, $\chi_=[T(K_{1,n})]=n+1,\forall n\ge 3$ and $n\in N$.

Remark: The equitable chromatic number of middle graph and total graph of star graph is the same.

References

1. Akbar Ali.M.M., Kaliraj.K., Vernold Vivin.J., On equitable coloring of Centralgraphs, And Total Graphs, Discrete Mathematics, 33, (2009), 1-6.
2. J. A. Bondy and U. S. R. Murty, Graph theory with Applications, London, MacMillan 1976.
3. Frank Harary, Graph Theory, Narosa Publishing home – 1969.
4. Graph coloring – Wikipedia, the free encyclopedia.
5. Hanna Furmaczyk and Marek Kubale, The Complexity of Equitable Vertex Coloring of Graphs, Journal of Applied Computer Science, 13(2) (2005), 95 -107.
6. Ko-WeiLih and Pou-Lin Wu, On equitable coloring of bipartite graphs, Discrete Mathematics, 151, (1996),155-160.
7. K.W. Lih, The equitable coloring of graphs, Handbook of Combinatorial Optimization, 3, Kluwer Acad. Publ., Boston, MA, (1998), 543-566.
8. Ko-WeiLih – Recent progress in Equitable coloring of Graphs, October 27 -2010.
9. W. Meyer, Equitable coloring Amer.Math.Monthly, 80 (1973), 920-922.
10. D. Michalak, On Middle and Total graphs with coarseness number equal 1, Springer Verlag Graph theory, Lagow proceedings, Berlin New York (1981), 139-150.
11. Sriram Pemmaraju, Kittikoran Nakprasit and Alexandr Kostochka, equitable colorings with Constant Numbers of Colors, Proceedings of the Fourteenth ACM-SIAM Symposium On Discrete Algorithms (SODA) , 2003.
12. Vernold Vivin.J, Akbar Ali.M.M and K.Thilagavathi, On Harmonius coloring of Central graphs, Advances and Applications in Discrete Mathematics, 2(1) (2008), 17-33.