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A Few Inherent Attributes of One Dimensional Nonlinear Map

Tarini Kr. Dutta[1], Jayanta Kr. Das[2], Anil Kr. Jain[3]

[1]Professor, Department of Mathematics, Guwahati University, Guwahati, Assam - 781014 INDIA.

[2]Assistant Professor, Department of Mathematics, K.C. Das Commerce College, Guwahati, Assam - 781008, INDIA.

[3]Assistant Professor, Department of Mathematics, Barama College, Barama, Assam - 781346, INDIA.

Corresponding Addresses:

[1]tkdutta2001@yahoo.co.in, [2]dipalidas371@gmail.com, [3]jainanil965@gmail.com

Research Article

Abstract: Here, we consider a one dimensionalnonlinear cubic map to find out a few inherent attributes i.e.fixed points, periodic points, bifurcation values of periods$2^n, n = 0,1,2,3,4 \ldots\ldots\ldots\ldots\ldots$...We use suitable numerical methods and have shown how the period doubling bifurcation points ultimately converge to the Feigenbaum constant. We have calculated Feigenbaum δvalue also. We have further verified our findings with the help of bifurcation diagram, Lyapunavexponent, time series analysis of the map. Computer software package 'Mathematica' and 'C-program' are used prudentially to implement numerical algorithms for our purpose.

Keywords: Fixed points, Periodic points, Bifurcation points, Feigenbaum constant, Lyapunov exponent, Time series.

1. Introduction

The nonlinear dynamics of physical systems are analyzed by obtaining discrete models. Mathematically, models are represented by maps. Maps arise in various ways. The ways are--------

i) As tools from analyzing differential equations.

ii) As models of natural phenomenon.

iii) As simple examples of chaos.

Maps are capable of more varied behavior than differential equations because the points jump discontinuously along their orbits rather than flow continuously. [5]

In this paper, we have considered the model$f(x) = x + \mu x(1 - x^2)$ where $x \in [0, 1]$ and $\mu \in]0, 2]$ is an adjustable parameter. Here a detailed analysis of period doubling bifurcation of model has been discussed. Here we shall also study some associated universalities, particularly the route from order to chaos, as developed by Mitchel's J Feigenbaum an American physicist.

Secondly, we have determined the accumulation point and draw the bifurcation graph of the model and verify that chaos occur beyond accumulation point.

Thirdly, the graph of Lyapuonv exponent confirms about existence of chaotic region.

Fourthly, the graphs of time series analysis are confirmed in order to support our periodic orbits of periods $2^0, 2^1, 2^2 \ldots\ldots\ldots$ [6, 7, 8]

2. Fixed point:-[7, 8]

Let Xbe a topological space and $f: X \to X$ be a map. A real number $x^*$is called a fixed point of the function f iff$(x^*) = x^*$. Our model is $f(x) = x + \mu x(1 - x^2)$ where $x \in [0,1]$ and $\mu \in]0,2]$. Clearly the solution $f(x^*) = x^*$gives the fixed point of f. A fixed point $x^*$is said to be a

i) Stable fixed point or attractor if$|f^/(x^*)| < 1$.

ii) Unstable fixed point or repelled if$|f^/(x^*)| > 1$.

iii) Super attractive or super stable if $f^/(x^*) = 0$.

The physical significance of a 'fixed point' is that it can be thought as an 'equilibrium point'. [16]

3. Periodic point and Periodic orbit

Let Xb e a topological space and $x \in X$. For any $m \in Z_+$, we say x is a periodic point orperiod m point if $f^m(x) = x$and$f^j(x) \neq x$ for $j = 1,2,3 \ldots\ldots\ldots\ldots (m-1)$. Underthis circumstance the orbit of x is called a periodic orbit or period m orbit. Also we say that m is the period of the periodic point or periodic orbit.

Note: A fixed point can be included under this definition so periodic points of period one. Conversely, a periodic point of period $m$of a map f can be viewed as the fixed point(s) of mth degree iteration of themap.

4. Bifurcation& Bifurcation point[17]

Fixed points can be created or destroyed, or their stability can be changed. These qualitative changes in the dynamic are called bifurcations & the parameter values at which they occur are called bifurcation points.

5.Chaos[10]

Chaos is the term used to describe the apparently complex behavior of what we consider to be sample, well behaved systems. Chaotic behavior, when we looked at casually, looks erratic and almost random--almost like the behavior of a system strongly influenced by outside, random 'noise' or complicated behavior of a system with many, many degrees of freedom, each 'doing its own thing"

Also, some sudden and dramatic changes in nonlinear systems may give rise to the complex behavior called chaos. The noun chaos and adjective chaotic are used to describe the time behavior of a system. When that behavior is a periodic (it never exactly repeats) and is apparently random or hoist.

6. Our Study on ourmodel

a)**Finding maximum value of f(x) and its range:**

In our model, at$x = +\sqrt{\frac{1+\mu}{3\mu}}$,$f(x)$is maximum as $f^{//}(x) < 0$for $\mu > 0$ and at this point maximum value for$f(x)$ is$\frac{2(1+\mu)^{\frac{3}{2}}}{3\sqrt{3\mu}}$. Again if take $\mu = 2$, then we get f(x) is maximum at $x = \frac{1}{\sqrt{2}}$and this maximum value is1.414 as $x > 0$. Hence the range may be taken as$[0, 1.414]$.

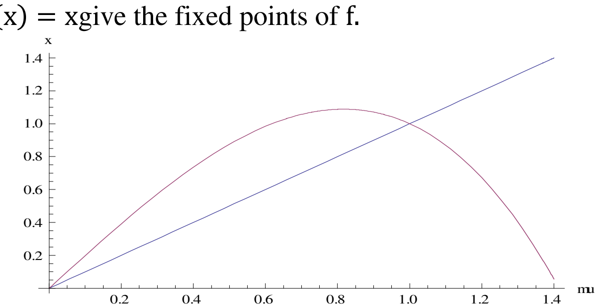
b) **Finding fixed points & our interest:**

Solving $f(x) = x$, we get $x = -1, 0, 1$

Therefore, the fixed points are$-1, 0, 1$and out of all these points our interesting points are 0,1 as$x \in [0,1]$.

c) **Finding the first bifurcation point:**

By stability criterion, we know that the fixedpoint $x = x^*$ is stable one if$\left|f^{/}(x)\right|_{x=x^*} < 1$, otherwise unstable.In case of our model, $f^{/}(x) = 1 + \mu - 3\mu x^2$ and $\left|f^{/}(x)\right|_{x=0} = 1 + \mu > 1$ for $\mu > 0$, therefore$x = 0$ is an unstable fixed point.Again $\left|f^{/}(x)\right|_{x=1} = 1 - 2\mu$.This absolutevalue remains less than 1 for$0 < \mu < 1$and greater than 1 for $\mu > 1$.Therefore, the fixed point $x = 1$ is astable onefor$\mu \in (0\,, 1)$and unstable onewhen $\mu > 1$.Hence $\mu = 1$ is the first bifurcationpointof our model and theintersection points of$f(x) = x + \mu x(1 - x^2)$and $f(x) = x$give the fixed points of f.



***Figue* 6.1**: Graphs of $f(x) = x + \mu x(1 - x^2)andf(x) = x$ at the parameter $\mu = 1.000000000$.

(d) **Finding the 2nd bifurcation point and theothers**:-

For the second bifurcation point we consider theiterated map $f^2(x)$and the fixed pointsof it are given by solving the equation$f^2(x) = x$, which is a ninth degree equation, so give nine roots.

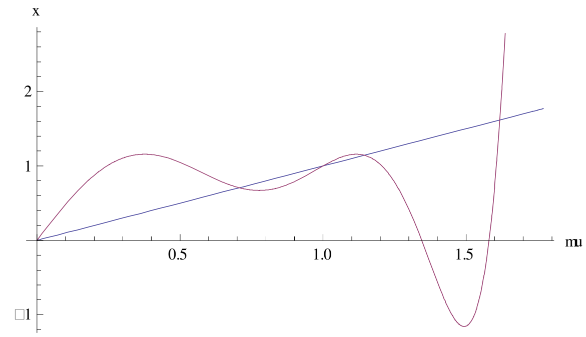


Figure 6.2 Graph of $f^2(x)$and $f(x) = x$ at the parameter $\mu = 1.23606797749978958$0and their intersection gives four fixed points of$f(x)$.

With the help of C-programming and using Newton-Rapson method we get the second bifurcation point $\mu_2 = 1.236067749978950$and the corresponding periodic point is $x = 0.707107$

By the same way we get the other bifurcation points andalso given one graph in this regard---------

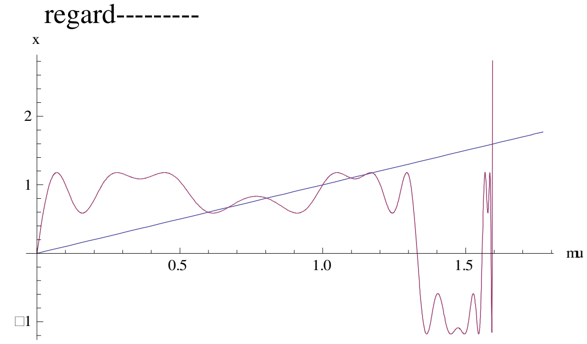


***Figure* 6.3**: Graph of $f^4(x)$and $f(x) = x$ at the parameter $\mu = 1.299227939650204000$and their intersection gives the fixed points off^4.

7. Numerical algorithm to find periodic points, derivatives of different iterates of our map and the bifurcation points

Newton-Recurrence formula is------

$x_{n+1} = x_n - \frac{g(x_n)}{[\frac{d}{dx}g(x_n)]}$, $n = 1,2,3,4 \ldots \ldots$ .With the help of this formula we can find out ourfixed points.

Let the initial value of x be x_0. Then

$$f(x_0) = x_0 + \mu x_0(1 - x_0^2) = x_1(say)$$

$$f^2(x_0) = f(x_1) = x_1 + \mu\, x_1(1 - x_1^2) = x_2(say)$$

Proceeding in this way we get the recurrence formula as follows---

$$x_n = x_{n-1} + \mu x_{n-1}(1 - x_{n-1}^2) \quad , \quad n = 1,2,3,4, \ldots \ldots \ldots \ldots \ldots$$

Now, the derivative off^k can be defined as follows-----------

$\left|\frac{df}{dx}\right|_{x=x_0} = 1 + \mu - 3\mu x_0^2$. And by chain rule of different ion we get

$\left|\frac{df^2}{dx}\right|_{x=x_0} = \left|\frac{df}{dx}\right|_{f(x_0)} \left|\frac{df}{dx}\right|_{x=x_0} = (1+\mu-3\mu x_1^2)(1+\mu-3\mu x_0^2), \quad x_1 = f(x_0)$

Proceeding in this way we get

$$\left|\frac{df^k}{dx}\right|_{x=x_0} = \left(1+\mu-3\mu x_{k-1}^2\right)\left(1+\mu-3\mu x_{k-2}^2\right) - - - - - (1+\mu-3\mu x_0^2)\ .$$

We always remember that the bifurcation value for the map f^k where its derivative $\frac{df^k}{dx}$ at periodic point is equal to -1 . Now it is given below a table of the bifurcation points and one of the fixed points(periodic points) at the corresponding bifurcation point and Feigenbaum delta value is

$$\delta_n = \frac{\mu_{n+1}-\mu_n}{\mu_{n+2}-\mu_{n+1}}$$

Numerical calculation of bifurcationpoints

Bifurcation Points	One of the periodic points	Feigenbaum delta value (Experimental value)
$\mu_1 = 1.0000000000000000$	1.000000	
$\mu_2 = 1.2360679774997895$	0.707107	
$\mu_3 = 1.2880317544828430$	0.609182	$\delta_1 = 4.5429338947968838 46$
$\mu_4 = 1.2992279396502040$	0.587954	$\delta_2 = 4.641203785601336114$
$\mu_5 = 1.3016289140370815$	0.583623	$\delta_3 = 4.663183925890289113$
$\mu_6 = 1.3021432715814591$	0.583614	$\delta_4 = 4.667809342385706860$
$\mu_7 = 1.3022534595730905$	0.589555	$\delta_5 = 4.667901748491901825$
$\mu_8 = 1.3022770594856267$	0.589465	$\delta_6 = 4.668030799697742375$
$\mu_9 = 1.3023275854537916$	0.589471	$\delta_7 = 4.669200000007389709$

From the above table we conclude that theFeigenbaumdelta converge to 4.669200- - -- -- -- and the following bifurcation diagram indicates the universal route to chaos for our model.

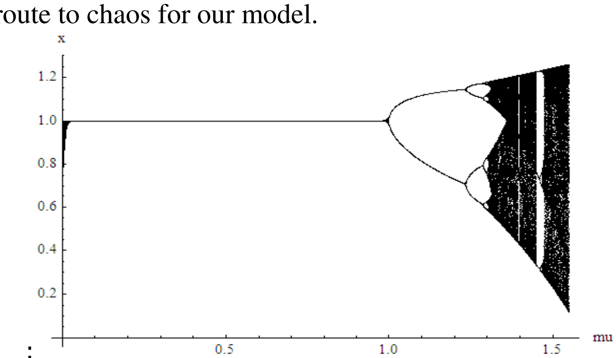


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***Figure*7.1**: Bifurcation diagram of $f(x) = x + \mu x(1 - x^2)$

8. Accumulation point: [6, 7, 8]

Since our model follows a period doublingbifurcation, therefore we can consider that $\{\mu_n\}$ be the sequence ofbifurcation points.With the help of Feigenbaum delta(δ), if we know first(μ_1)and second(μ_2)bifurcation points, then wegetμ3 $\mu_3 \approx \frac{\mu_2-\mu_1}{\delta} + \mu_2 \ldots\ldots\ldots$ (i) . Similarly we get $\mu_4 \approx \frac{\mu_3-\mu_2}{\delta} + \mu_3 \ldots\ldots\ldots$ (ii).

From (i), (ii) we get $\mu_4 \approx (\mu_2-\mu_1)\left(\frac{1}{\delta}+\frac{1}{\delta^2}\right) + \mu_2$. If we go on this procedure to calculate μ_5 , μ_6 and so on, we just obtain more terms in the sum involving powers of $\left(\frac{1}{\delta}\right)$ .We acknowledge this sum as a geometricseries and after simplification we obtain the result. [10]

$\mu_\infty \approx \frac{\mu_{n+1}-\mu_n}{\delta-1} + \mu_{n+1} \ldots\ldots\ldots$ (iii) .The expression (iii) is exact when the bifurcation ratio $\delta_n = \frac{\mu_{n+1}-\mu_n}{\mu_{n+2}-\mu_{n+1}}$ is equal $\forall n$ and then $\lim_{n\to\infty}\delta_n = \delta$. Hence $\{\mu_{\infty,n}\}$ is the sequence and $\lim_{n\to\infty}\mu_{\infty,,n} = \mu_\infty$ .Using the experimental bifurcation points the sequence of accumulation points $\{\mu_{\infty,n}\}$are calculated for some values of n and the points are mentioned under in this regard ----------

$\mu_{\infty,1}$ = 1.302698633949827520026682583905 6

$\mu_{\infty,2}$ = *1.302302797808826969632032447497 8*

$\mu_{\infty,3}$ = 1.302314347789645369468703373194 3

$\mu_{\infty,4}$ = 1.302325033856056056545835437165 5

$\mu_{\infty,5}$ = 1.323127321481359127721350581629 7

$\mu_{\infty,6}$ = 1.30230067760396197979 51 50636487

The above sequence converge to the value 1.3023 - - - - - , which is the required accumulation point.

9. Lyapunov Exponent

The Lyapunov exponent is an experimental device. This device isstrong and efficacious. It has ability to separate unstable, chaotic behavior from that which is stable and predictable.With the help of it we can measure these properties also. All the chaotic systems definitely have the phenomenon of sensitive dependence on initial conditions (or perturbations of the orbit). Out of all signatures one of the signatures of chaos is the divergence of adjacent (nearby) trajectories. The adjacent trajectories diverge exponentially on strange attractors. Lyapunov exponent quantifies theexponentially divergence of two trajectories starting very close to each other. This exponent has two types…….

(i) The first one is positive Lyapunov exponent. It indicates the exponential divergence of the trajectory which confirms chaos.

(ii) The second isnegative Lyapunov exponent. This is associated with regular behavior (periodic orbit). [4]

The Lyapunov exponent(L)computed using the derivative method is defined by

$L = \lim_{n\to\infty}\frac{1}{n}\left(\log\left|f_\mu^1\ (x_1)\right| + \log\left|f_\mu^1\ (x_2)\right| + - - - - - + \log\left|f_\mu^1\ (x_n)\right|\right)$.Here f_μ^1represents differentiation with respect to x and $x_1, x_2, x_3, - - - -, x_n$ are successive iterates.[15]

In this paper, Lyapunov exponent is calculated, to verify how much accurate are the accumulation points.

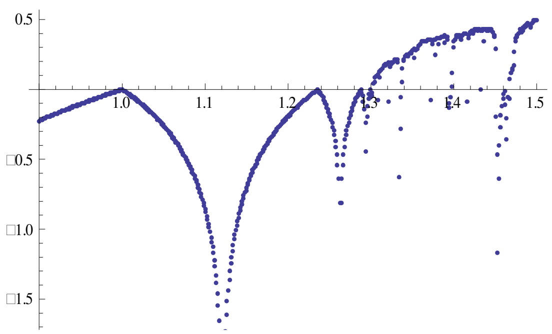

***Figure* 9.1**: Graph of Lyapunov Exponent for parameter from *0* to *1.5*

From the graph of Lyapunov experiment, we see that some portions lie in the negative side of the parameter axis indicating regular behavior (periodic orbit) and the portions lie on the positive side of parameter axis confirm us about the assistance of chaos for a model.

10. Time series analysis[5]

It is a type of plot. It is frequently used for visualization of the solution of one dimensional difference equation of the form $x_{n+1}=f(x_n)$, $n=0,1,2,3,4\ldots\ldots\ldots$given by a map called the time series. It consists of a representation of the variable x_n as a function of n. Typically the horizontal axis representsn and the vertical axis represents x_n In case of the map we have considered, the difference equation is given by $x_{n+1}=x_n+\mu x_n(1-x_n^2)$, $n=0,1,2,3,4\ldots\ldots$

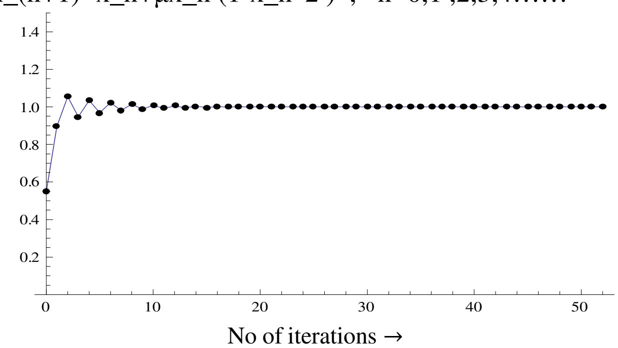

***Figure* 10.1**: Time series graph showing period *1* behavior for$\mu = 0.99$

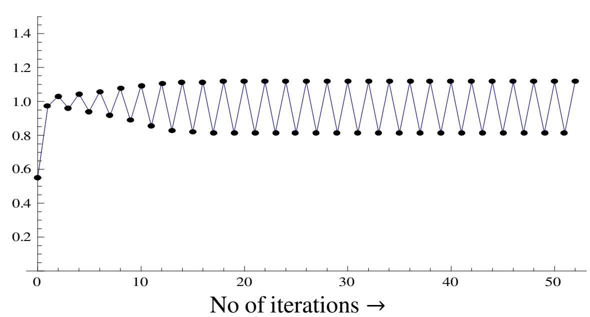

***Figure* 10.2**: Time series graph showing period 2 behavior for $\mu = 1.1$

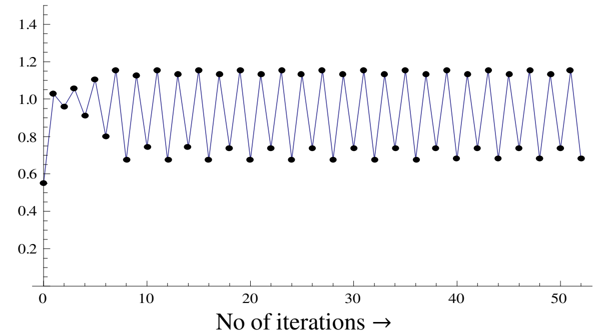

***Figure* 10.3**: Time series graph showing period 4 behavior for $\mu = 1.24$

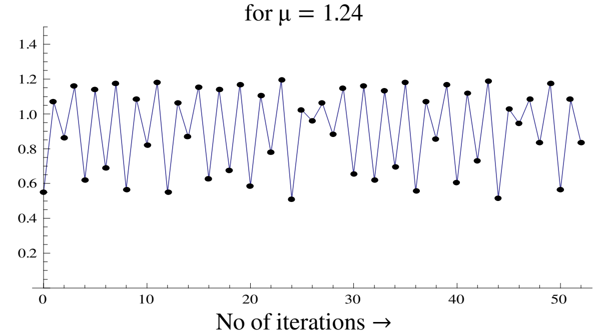

***Figure* 10.4**: Time series graph showing chaotic behavior for $\mu = 1.35$

The set forwards which the x values converge is called an attractor. The above figures show that an attractor may be a fixed point, a limit cycle (or a periodic attractor) or a chaotic attractor. In case of the map, we have considered, if we start with a value of μ less than 1, successive points 'flow' to a fixed point at a non-zero value of □indicating period one behavior inFig 10.1. However, for values of μ slightly greater than 1 the fixed point 'bifurcates' to a 'limit cycle' of period 2 in Fig 10.2. This then bifurcates again i.e. the period double at a larger value of □ to a limit cycle with period 4 in Fig 10.3. As μ increase, the period continues to double at successively closer and closer value ofμuntil we have chaotic behavior in Fig 10.4. In case of the map we have considered,this is illustrated in the above figures with four values ofμ($\mu = 0.99$,period *1* fixed point), ($\mu = 1.1$,period 2 limit cycle), ($\mu = 1.24$, period 4 limit cycle) and ($\mu = 1.35$,chaos).

References

1. Arrowsmith, D.K. and Place, C.M., An Introduction to Dynamical Systems, Cambridge University Press, (1994).
2. Denny, G., Encounters with Chaos, McGraw-Hill, Inc. (1992).
3. Devaney, R.L., An Introduction to chaotic dynamical system, 2th edition, Addison Wesley, (1989).
4. Das, N. Some Aspect of Bifurcations and Chaos on Nonlinear Population Models, Ph.D. Thesis, Department of Mathematics, Gauhati University, (2012)
5. Dutta, N., Bifurcation and Chaos in Non-linear Population Dynamics and their Statistical Analysis, Ph.D. Thesis, Department of Mathematics, Gauhati University, (2012).

6. Dutta, T.K. and Bhattacharjee, D., Bifurcation, Lyapunove Exponentand fractal Dimension in a non-linear Map (2010 AMS Classification: 37G15, 37G35, 3745.
7. Dutta, T.K., Jain, A.Kr. and Bhattacharjee, D., "Period Doubling Scenario in a One Dimensional Nonlinear Map", international J. Math. Sci & Engg. Appls (IJMSEA
8. Dutta, T.K., Jain, A.Kr. and Bhattacharjee, D.,"Some Aspects of Dynamical Behavior in a One Dimensional Non Linear Chaotic Map"
9. Hacibekirogul, G., Caglar Mand Polatoglu, Y., The higher order Schwarzian derivative: Its application for chaotic behavior and new invariant sufficient condition for chaos, nonlinear analysis: Real World Application, 10(3), 1275-2008.
10. Hilborn, R.C., Chaos and nonlinear Dynamics: An Introduction for Scientists and Engineers, Oxford University Press, (1994).
11. Hirsch, M.W. Stability and convergence in strongly monotone dynamical systems, J. reinedanew. Math., 383 (1988), 1-53.
12. Hussein, H.J.A. and Abed, F.S., On Some Dynamical Properties of Unimodel maps Pure Mathematical Science, 1(2) (2012), 53-65.
13. Gutierrez, J.M. and Iglesias, A., Mathematica Package for analysis and control of Chaos in nonlinear systems, Computer in Physics, 12(6) (Nov/Dec 1998).
14. Lynch Stephen. Dynamical Systems with Applications using Mathematica.
15. Mathews, J.H.., Numerical Methods for Mathematics, Science and Engineering, Prentice Hall, (1994).
16. Sharma, H. K. and Paul, R," Bifurcation Points in a Non Algebraic map" international J. Math. Sci & Engg. Appls (IJMSEA).
17. Strogatz, S. H. Nonlinear Dynamics and Chaos. Cambridge MA: Perseus, 1994.