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Bounds for the Differences of Moments of k-Records from a Continuous Distribution

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Research Article

Abstract: In this paper, we obtain the bounds for the differences of moments of k-records from a continuous population.

Key Words: k-Record Values, Orthogonal Inverse Expansion Method, Legendre polynomials.

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1. Introduction

Consider a sequence $\{X_n, n \geq 1\}$ of i.i.d r.v's with a common continuous d.f F. Let $X_{r,n}$ be the r^{th}order statistic among X_1, $X_2, X_3, \ldots, X_n$. For a fixed integer k≥ 1, define the k^{th} record times by

$$L(1,k) = k, L(n+1,k) = min(j{:}\, j > L(n,k), X_j > X_{j-k,j-1}), n \geq 1.$$

Then $X(n,k) = X_{L(n,k)-k+1,L(n,k)}$ is called the n^{th} $k-$ record value. Alternatively, the sequence $\{X(n,k), n \geq 1\}$ can also be obtained from $\{X_{n-k+1,n}, n \geq k\}$ by considering the variables which are distinct (Nevzorov, 2000). It is known (Nevzorov, 2000, p. 93) that $\{X(n,k), n \geq 1\}$ *and* $\left\{H\left(\frac{W_1+W_2+W_3+\cdots+W_n}{k}\right), n \geq 1\right\}$ are identically distributed.

where $W_1, W_2, \ldots, W_n$ are i.i.d unit exponential r.v's, $\mathrm{H}(x) = F^{-1}(1-e^{-x})$ and $F^{-1}(x) = \inf(y : F(y) \geq x)$. This gives

$$\{R(X(n,k)), n \geq 1\} = \{\textstyle\sum_{i=1}^{n} Y_i,\ n \geq 1\} \text{ distributionally} \tag{1.1}$$

where $Y_1, Y_2, Y_3, \ldots, Y_n$ are i.i.d $G(k,1)$ r.v's and $R(x) = -log(1-F(x))$.

Lemma 1.1: If $E|X|^{\gamma} < \infty$ then $E|X(n,k)|^{\delta} < \infty$ for all $\delta < \gamma$ and all $n \geq 1$.

Proof: Let $\mathrm{U}(n,k)$ be the n^{th} k- record associated with the standard uniform distribution. Then $X(n,k) = F^{-1}(\mathrm{U}(n,k))$. From (1.1) it follows that $-log(1-\mathrm{U}(n,k))$ has gamma distribution with parameters k and n. Therefore, the p.d.f of $\mathrm{U}(n,k)$ is

$$\frac{k^n}{(n-1)!}\{-log(1-x)\}^{n-1}(1-x)^{k-1}, 0 < x < 1.$$

Hence $E|X(n,k)|^{\delta} = \int_0^1 \frac{k^n}{(n-1)!}|F^{-1}(x)|^{\delta}\{-log(1-x)\}^{n-1}(1-x)^{k-1}dx$

$$= \frac{k^n}{(n-1)!}\int_0^1 (|F^{-1}(x)|^{\gamma})^{\frac{\delta}{\gamma}}((-\log(1-x))^{\frac{(n-1)\gamma}{\gamma-\delta}}(1-x)^{\frac{(k-1)\gamma}{\gamma-\delta}})^{1-\frac{\delta}{\gamma}}dx$$

$$\leq \frac{k^n}{(n-1)!}\left(\int_0^1 |F^{-1}(x)|^{\gamma}dx\right)^{\frac{\delta}{\gamma}}\left(\int_0^1\{-log(1-x)\}^{\frac{(n-1)\gamma}{\gamma-\delta}}(1-x)^{\frac{(k-1)\gamma}{\gamma-\delta}}dx\right)^{1-\frac{\delta}{\gamma}}$$

By Holder's inequality. The right side is finite since $\int_0^1 |F^{-1}(x)|^{\gamma}dx = E|X|^{\gamma} < \infty$ by hypothesis and

$$\int_0^1\{-\log(1-x)\}^{\frac{(n-1)\gamma}{\gamma-\delta}}(1-x)^{\frac{(k-1)\gamma}{\gamma-\delta}}dx = \frac{\Gamma\left(\frac{(n-1)\gamma}{\gamma-\delta}+1\right)}{\left(\frac{k\gamma-\delta}{\gamma-\delta}\right)^{\frac{(n-1)\gamma}{\gamma-\delta}+1}} \text{ is finite.}$$

We shall assume that $E|X_1|^{2\alpha} \prec \infty$ for some integer $\alpha \succ 0$ so that $E|X_1|^{\alpha} \prec \infty$ and $E|X(n,k)|^{\alpha} \prec \infty$. Let r be any integer such that $0 \prec r \leq 2\alpha$. Put $\mu_r^{'} = EX_1^r$.

Note that

$$EX^{\alpha}(n,k) = \int_0^1 \frac{k^n}{(n-1)!} U(u)^{\alpha} \{-log(1-u)\}^{n-1}(1-u)^{k-1} du \tag{1.2}$$

where $U(u) = F^{-1}(u)$.

Sugiura (1962, 1964) developed a method called the 'Orthogonal Inverse Expansion Method' for the evaluation of bounds and obtaining approximations for the means, variances and covariances of order statistics. This was generalized by Castillo (1988). Castillo obtained the bounds for the α^{th} raw moments of order statistics and also the bounds for the differences of moments. Arnold et al.,(1998) obtained bounds for the means of record values. In section 2, the orthogonal inverse expansion method is explained. The bounds for the differences moments of k- record values are obtained in section 3. Section 4 deals with the computation of the Fourier coefficients. The last section is devoted for the application of the results when the orthonormal system of the orthogonal inverse expansion method is the sequence of Legendre polynomials and the basic distribution is standard normal.

2. The Orthogonal Inverse Expansion Method.

Let the class of square integrable functions defined on (0,1) be denoted by $L^2(0,1)$. Let $\{\psi_i, i \geq 0\}$ be a complete orthonormal system of functions in $L^2(0,1)$ where $\psi_0 = 1$. That is,

$$\int_0^1 \psi_i(u)du = 0, \int_0^1 \psi_i^2(u)du = 1 \ and \ \int_0^1 \psi_i(u)\psi_j(u)du = 0, i \neq j, i, j = 1,2,....... \tag{2.1}$$

Then is it is known that for any function f in $L^2(0,1)$,

$$f(u) \equiv \lim_{m \to \infty} \sum_{i=0}^{m} a_i \psi_i(u) \ and \ \int_0^1 f^2(u)du = \sum_{i=0}^{\infty} a_i^2 \tag{2.2}$$

$$where \quad a_i = \int_0^1 f(u)\psi_i(u)du \ , i = 0,1,2,3,....$$

Here a_i is known as the Fourier coefficient of f. Let f and g be any two functions in $L^2(0,1)$ with respective Fourier coefficients $a_i \ and \ b_i$. For any

Integer $m \geq 0$, we have,

$$\left| \int_0^1 f(u)g(u)du - \sum_{i=0}^{m} a_i b_i \right|$$

$$= \left| \int_0^1 \left\{ f(u) - \sum_{i=0}^{m} a_i \psi_i(u) \right\} \left\{ g(u) - \sum_{i=0}^{m} b_i \psi_i(u) \right\} du \right| \quad from \ (2.1).$$

$$\leq \left\{ \int_0^1 f^2(u) - \sum_{i=0}^{m} a_i^2 \right\}^{1/2} \left\{ \int_0^1 g^2(u)du - \sum_{i=0}^{m} b_i^2 \right\}^{1/2} \tag{2.3}$$

by the Cauchy-Schwarz inequality and (2.1).

Remark 2.1: From (2.2), it follows that the R.H.S of (2.3) tends to zero as $m \to \infty$. Hence for large m, $\sum_{i=0}^{m} a_i b_i$ is a reasonable approximation for $\int_0^1 f(u)g(u)du$. Also, $\sum_{i=0}^{m} a_i b_i \pm R$ where R is the R.H.S of (2.3) are the bounds for $\int_0^1 f(u)g(u)du$. The accuracy of the approximation as well as the bounds increases as m increases.

3. The Bounds

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. r.v's with a common continuous d.f F. Let $\{X(n,k), n \geq 1\}$ be the sequence of corresponding k-record values. Assume that $E|X_1|^{2\alpha} \prec \infty$ for some positive integer α. Let $\{\psi_i, i \geq 0\}$ be a complete orthonormal system of functions in $L^2(0,1)$.

Theorem 3.1: Let $\alpha(\geq 1)$ and k (≥ 1) be any fixed integers. For any integers $n_1(\geq 1)$, $n_2(>1)$ and $\mathrm{m} \geq 0$ we have

$$|\mathrm{EX}^\alpha(\mathrm{n}_2,\mathrm{k}) - \mathrm{EX}^\alpha(\mathrm{n}_1,\mathrm{k}) - \textstyle\sum_{i=0}^{m} a_i\, b_i(n_1,n_2,k)\,| \leq$$

$$\left(\mu'_{2\alpha} - \textstyle\sum_{i=0}^{m} a_i^2\right)^{\frac{1}{2}} \left(\binom{2n_1-2}{n_1-1}\frac{k^{2n_1}}{(2k-1)^{2n_1-1}} + \binom{2n_2-2}{n_2-1}\frac{k^{2n_2}}{(2k-1)^{2n_2-1}} - 2\binom{n_1+n_1-2}{n_1-1}\frac{k^{n_1+n_1}}{(2k-1)^{n_1+n_1-1}} - \textstyle\sum_{i=0}^{m} b_i^2(n_1,n_2,k) \right)^{\frac{1}{2}} \qquad (3.1)$$

where $a_i = \int_0^1 F^{-1}(u)\psi_i(u)du$, $i = 0,1,2,3,....$ and

$$b_i(n_1,n_2,k) = \int_0^1 \left(\frac{k^{n_2}}{(n_2-1)!}\left(-log(1-u)\right)^{n_2-1} - \frac{k^{n_1}}{(n_1-1)!}\left(-log(1-u)\right)^{n_1-1} \right)(1-u)^{k-1}\psi_i\, du$$

In particular, for m=0 we have

$$|\mathrm{EX}^\alpha(\mathrm{n}_2,\mathrm{k}) - \mathrm{EX}^\alpha(\mathrm{n}_1,\mathrm{k})\,|$$

$$\leq \left(\mu'_{2\alpha} - {\mu'_\alpha}^2\right)^{\frac{1}{2}} \left(\binom{2n_1-2}{n_1-1}\frac{k^{2n_1}}{(2k-1)^{2n_1-1}} + \binom{2n_2-2}{n_2-1}\frac{k^{2n_2}}{(2k-1)^{2n_2-1}} - 2\binom{n_1+n_1-2}{n_1-1}\frac{k^{n_1+n_1}}{(2k-1)^{n_1+n_1-1}} \right)^{\frac{1}{2}}$$

Proof: Take $f(u) = F^{-1}(u)$ and

$$g(u) = \left(\frac{k^{n_2}}{(n_2-1)!}\left(-log(1-u)\right)^{n_2-1} - \frac{k^{n_1}}{(n_1-1)!}\left(-log(1-u)\right)^{n_1-1}\right)(1-u)^{k-1}$$

in (2.3). Then from (1.2) we have $\int_0^1 f(u)g(u)du = \mathrm{EX}^\alpha(\mathrm{n}_2,\mathrm{k}) - \mathrm{EX}^\alpha(\mathrm{n}_1,\mathrm{k})$. Also $\int_0^1 f^2(u)du = \mu'_{2\alpha}$.

Note that $\int_0^1 g^2(u)du = \int_0^1 \frac{k^{2n_2}}{((n_2-1)!)^2}\left(-log(1-u)\right)^{2n_2-2}(1-u)^{2k-2}du$

$$+ \int_0^1 \frac{k^{2n_1}}{((n_1-1)!)^2}\left(-log(1-u)\right)^{2n_1-2}(1-u)^{2k-2}du$$

$$- 2\int_0^1 \frac{k^{n_1+n_2}}{(n_1-1)!(n_2-1)!}\left(-log(1-u)\right)^{n_1+n_2-2}(1-u)^{2k-2}du$$

Putting 1-u =t and simplifying we get

$$\int_0^1 g^2(u)du = \binom{2n_1-2}{n_1-1}\frac{k^{2n_1}}{(2k-1)^{2n_1-1}} + \binom{2n_2-2}{n_2-1}\frac{k^{2n_2}}{(2k-1)^{2n_2-1}} - 2\binom{n_1+n_1-2}{n_1-1}\frac{k^{n_1+n_1}}{(2k-1)^{n_1+n_1-1}}.$$

Now (2.3) reduces to (3.1).

In particular, if m = 0, the result is immediate after noting that $a_0 = \mu'_\alpha$ and $b_0(n_1, n_2, k) = 0$.

Theorem 3.2: Assume that the common continuous distribution of the variables in the basic sequence is symmetric about the origin. Let $\alpha(\geq 1)$ and k (≥ 1) be any fixed integers. For any integers $n_1(\geq 1)$, $n_2(>1)$ and m ≥ 0 we have

$$|\mathrm{EX}^{\alpha}(n_2,k) - \mathrm{EX}^{\alpha}(n_1,k) - \sum_{i=0}^{m} a_i\, b_i^*(n_1,n_2,k)\,| \leq \left(\mu'_{2\alpha} - \sum_{i=0}^{m} a_i^2\right)^{\frac{1}{2}} \left(\binom{2n_1-2}{n_1-1}\frac{k^{2n_1}}{2(2k-1)^{2n_1-1}} + \binom{2n_2-2}{n_2-1}\frac{k^{2n_2}}{2(2k-1)^{2n_2-1}} - \binom{n_1+n_1-2}{n_1-1}\frac{k^{n_1+n_1}}{(2k-1)^{n_1+n_1-1}} + \frac{(-1)^{\alpha}k^{2n_1}A_{n_1k}}{2} + \frac{(-1)^{\alpha}k^{2n_2}A_{n_2k}}{2} - (-1)^{\alpha}k^{n_1+n_2}A_{n_{1}n_{2}k} - \sum_{i=0}^{m} b_i^{*2}(n_1,n_2,k)\right)^{\frac{1}{2}} \quad (3.2)$$

where $a_i = \int_0^1 F^{-1}(u)\psi_i(u)du$, $i = 0,1,2,3,....$,

$b_i^*(n_1,n_2,k) = \int_0^1 \left(g_{n_2}(u) - g_{n_1}(u)\right)\psi_i(u)du$,

$g_n(u) = \frac{k^n}{2(n-1)!}((-1)^{\alpha}(-logu)^{n-1}u^{k-1} + (-\log(1-u))^{n-1}(1-u)^{k-1})$,

$A_{n_k} = \frac{1}{((n-1)!)^2}\int_0^1 \left(log(1-u)\right)^{n-1}(logu)^{n-1}u^{k-1}(1-u)^{k-1}du$ and

$A_{n_1n_{2k}} = \frac{1}{(n_1-1)!(n_2-1)!}\int_0^1 \left(log(1-u)\right)^{n_1-1}(logu)^{n_2-1}u^{k-1}(1-u)^{k-1}du.$

In particular, for m = 0, we have ,

$$|\mathrm{EX}^{\alpha}(n_2,k) - \mathrm{EX}^{\alpha}(n_1,k)\,| \leq \left(\mu'_{2\alpha} - \mu_{\alpha}'^{\,2}\right)^{\frac{1}{2}} \left(\binom{2n_1-2}{n_1-1}\frac{k^{2n_1}}{2(2k-1)^{2n_1-1}} + \binom{2n_2-2}{n_2-1}\frac{k^{2n_2}}{2(2k-1)^{2n_2-1}} - \binom{n_1+n_1-2}{n_1-1}\frac{k^{n_1+n_1}}{(2k-1)^{n_1+n_1-1}} + \frac{(-1)^{\alpha}k^{2n_1}A_{n_1k}}{2} + \frac{(-1)^{\alpha}k^{2n_2}A_{n_2k}}{2} - (-1)^{\alpha}k^{n_1+n_2}A_{n_{1}n_{2}k}\right)^{\frac{1}{2}}.$$

Proof: Since the distribution is symmetric about the origin, we notice that

U(u) = -U(1-u), 0 < u < 1 where $U(u) = F^{-1}(u)$. Putting y = 1-u in (1.2) and simplifying we see that

$$EX^{\alpha}(n,k) = (-1)^{\alpha}\int_0^1 \frac{k^n}{(n-1)!} U(y)^{\alpha}(-logy)^{n-1}y^{k-1}dy \ .$$

Hence

$$EX^{\alpha}(n,k) = \frac{k^n}{2(n-1)!}\int_0^1 U(u)^{\alpha}((-1)^{\alpha}(-logu)^{n-1}u^{k-1} + (-\log(1-u))^{n-1}(1-u)^{k-1})du\ .$$

This gives, $\mathrm{EX}^{\alpha}(n_2,k) - \mathrm{EX}^{\alpha}(n_1,k) = \int_0^1 U(u)^{\alpha}\left(g_{n_2}(u) - g_{n_1}(u)\right)du$

where $g_n(u) = \frac{k^n}{2(n-1)!}((-1)^{\alpha}(-logu)^{n-1}u^{k-1} + (-\log(1-u))^{n-1}(1-u)^{k-1})$.

Taking f(u) = $U(u)^{\alpha}$ and g(u) = $g_{n_2}(u) - g_{n_1}(u)$ in (2.3) we see that

$\int_0^1 f(u)g(u)du = \mathrm{EX}^{\alpha}(n_2,k) - \mathrm{EX}^{\alpha}(n_1,k)$, $\int_0^1 f^2(u)du = \mu'_{2\alpha}$ and

$\int_0^1 g^2(u)du = \int_0^1 g_{n_1}^2(u)du + \int_0^1 g_{n_2}^2(u)du - 2\int_0^1 g_{n_1}(u)g_{n_2}(u)du.$

Now, $\int_0^1 g_n^2(u)du = \frac{k^{2n}}{4((n-1)!)^2}\Big(\int_0^1\left(-log(1-u)\right)^{2n-2}(1-u)^{2k-2}du + \int_0^1\left(-log(u)\right)^{2n-2}u^{2k-2}du +$

$2(-1)^{\alpha}\int_0^1\left(-log(1-u)\right)^{n-1}(1-u)^{k-1}\left(-log(u)\right)^{n-1}u^{k-1}du\Big).$

$= \frac{k^{2n}}{4((n-1)!)^2}\left(2\int_0^{\infty} e^{-(2k-1)t}t^{2n-2}dt + 2(-1)^{\alpha}\left((n-1)!\right)^2 A_{nk}\right)$

$= \frac{k^{2n}(2n-2)!}{2((n-1)!)^2(2k-1)^{2n-1}} + \frac{(-1)^{\alpha}k^{2n}A_{nk}}{2}$.

Also, $\int_0^1 g_{n_1}(u)\, g_{n_2}(u)du = \frac{k^{n_1+n_2}}{4(n_1-1)!(n_2-1)!}(\int_0^1(-\log(1-u))^{n_1+n_1-2}(1-u)^{2k-2}$

$+(-1)^{\alpha}\left(-log(1-u)\right)^{n_1-1}(-logu)^{n_2-1}(1-u)^{k-1}u^{k-1}$
$+(-1)^{\alpha}\left(-log(1-u)\right)^{n_2-1}(-logu)^{n_1-1}(1-u)^{k-1}u^{k-1}$
$+(-logu)^{n_1+n_1-2}u^{2k-2}$) du)
$=\frac{k^{n_1+n_2}}{2(n_1-1)!(n_2-1)!}\left(\int_0^{\infty} e^{-(2k-1)t}t^{n_1+n_1-2}dt+(-1)^{\alpha}(n_1-1)!\,(n_2-1)!\,A_{n_1n_2k}\right)$
since $A_{n_1n_2k}=A_{n_2n_1k}$.
$=\frac{k^{n_1+n_2}}{2(n_1-1)!(n_2-1)!}\binom{n_1+n_2-2}{n_1-1}+\frac{(-1)^{\alpha}k^{n_1+n_2}A_{n_1n_2k}}{2}$.
Hence $\int_0^1 g^2(u)du=\binom{2n_1-2}{n_1-1}\frac{k^{2n_1}}{2(2k-1)^{2n_1-1}}+\binom{2n_2-2}{n_2-1}\frac{k^{2n_2}}{2(2k-1)^{2n_2-1}}-\binom{n_1+n_1-2}{n_1-1}\frac{k^{n_1+n_1}}{(2k-1)^{n_1+n_1-1}}+\frac{(-1)^{\alpha}k^{2n_1}A_{n_1k}}{2}+\frac{(-1)^{\alpha}k^{2n_2}A_{n_2k}}{2}-(-1)^{\alpha}k^{n_1+n_2}A_{n_{1n_2k}}$.

Now (2.3) reduces to (3.2). In particular, when m = 0, the result is immediate after noting that $a_0=\mu'_{\alpha}$ and $b_0^*(n_1,n_2,k)=0$.

REMARK 3.1: Note that a_i depends only on F and is independent of n_1 , n_2 and k whereas both $b_i(n_1,n_2,k)$ and $b_i^*(n_1,n_2,k)$ depend only on n_1 , n_2 and k and are independent of F.

4. Computation of the Fourier Coefficients.

In this section, F stands for a general continuous d.f. We now derive explicit expressions for $a_i,\ b_i(n_1,n_2,k)\ and\ b_i^*(n_1,n_2,k)$ by taking the orthonormal system to be the system of Legendre polynomials defined by

$$\psi_i(u)=\frac{\sqrt{2i+1}}{i!}\frac{d^i}{du^i}\left\{u^i(u-1)^i\right\},0\prec u\prec 1,\ i=0,1,2,3,....$$

Consider $\psi_i(u)=\frac{\sqrt{2i+1}}{i!}\frac{d^i}{du^i}\left\{u^i(u-1)^i\right\},0\prec u\prec 1,\ i=0,1,2,3,....$

$$=\frac{\sqrt{2i+1}}{i!}\frac{d^i}{du^i}\left(u^2-u\right)^i=\frac{\sqrt{2i+1}}{i!}\frac{d^i}{du^i}\sum_{j=0}^{i}(-1)^j\binom{i}{j}u^{2i-j}=\sqrt{2i+1}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\binom{2i-j}{i}u^{i-j}$$

$$=\sqrt{2i+1}\sum_{j=0}^{i}(-1)^j\binom{i}{i-j}\binom{2i-j}{i-j}u^{i-j}=\sqrt{2i+1}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\binom{i+j}{j}u^j\ .$$

Hence

$$\psi_i(u)=\sqrt{2i+1}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\binom{i+j}{j}u^j \tag{4.1}$$

Also,

$$\psi_i(u)=\frac{\sqrt{2i+1}}{i!}\frac{d^i}{du^i}\left\{u^i(u-1)^i\right\}=\frac{\sqrt{2i+1}}{i!}\frac{d^i}{du^i}\left\{(1-u)^i(1-u-1)^i\right\}$$

$$=\frac{\sqrt{2i+1}}{i!}\frac{d^i}{du^i}\sum_{j=0}^{i}\binom{i}{j}(-1)^j(1-u)^{2i-j}=\sqrt{2i+1}\sum_{j=0}^{i}(-1)^{i+j}\binom{i}{j}\binom{2i-j}{i}(1-u)^{i-j}$$

$$=\sqrt{2i+1}\sum_{j=0}^{i}(-1)^{i+j}\binom{i}{i-j}\binom{2i-j}{i-j}(1-u)^{i-j}=\sqrt{2i+1}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\binom{i+j}{j}(1-u)^j$$

Thus we have

$$\psi_i(u)=\sqrt{2i+1}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\binom{i+j}{j}(1-u)^j \tag{4.2}$$

From (4.1) and (4.2) we observe that

$$\psi_i(u) = (-1)^i \psi_i(1-u),\ 0 \prec u \prec 1,\ i = 0,1,2,3,....$$

$$Let \quad a_i = \int_0^1 \{F^{-1}(u)\}^\alpha \psi_i(u)du,\ 0 \prec u \prec 1,$$

$b_i(n,k) = \int_0^1 \frac{k^n}{(n-1)!}\{-log(1-u)\}^{n-1}(1-u)^{k-1}\psi_i(u)\,du,$

$b_i(n_1,n_2,k) = \int_0^1 \left(\frac{k^{n_2}}{(n_2-1)!}\left(-log(1-u)\right)^{n_2-1} - \frac{k^{n_1}}{(n_1-1)!}\left(-log(1-u)\right)^{n_1-1}\right)(1-u)^{k-1}\psi_i\,du$

and

$b_i^*(n_1,n_2,k) = \int_0^1 \left(g_{n_2}(u,k) - g_{n_1}(u,k)\right)\psi_i(u)$ du

where

$$g_n(u,k) = \frac{k^n}{2(n-1)!}((-1)^\alpha(-logu)^{n-1}u^{k-1} + (-\log(1-u))^{n-1}(1-u)^{k-1})$$

and $k(\geq 1),\ \alpha(\geq 1),\ n\ (\geq 1),\ n_1\ ,\ n_2\ \left(1 \leq n_1 \prec n_2\right)$ are integers.

Theorem 4.1: (a) We have,

$$a_i = \sqrt{2i+1}\ \sum_{j=0}^{i} \frac{(-1)^{i-j}}{j+1}\binom{i}{j}\binom{i+j}{j} EY^{\alpha}_{j+1,\,j+1},\ i = 0,1,2,3,... \tag{4.3}$$

where $Y_{i+1,i+1}$ is the maximum in a random sample of size $i+1$ drawn from F . If F is symmetric about the origin and $\alpha + i$ is odd then $a_i = 0.$

$$(b) \quad b_i(n,k) = k^n\sqrt{2i+1}\ \sum_{j=0}^{i} \frac{(-1)^j}{(k+j)^n}\binom{i}{j}\binom{i+j}{j},\ i = 0,1,2,3,.... \tag{4.4}$$

$$(c) \quad b_i(n_1,n_2,k) = b_i(n_2,k) - b_i(n_1,k) \tag{4.5}$$

and

$$\text{(d)} \quad b_i^*(n_1,n_2,k) = \frac{\{1+(-1)^{\alpha+i}\}b_i(n_1,n_2,k)}{2} \tag{4.6}$$

Proof: (a). Note that from (4.1) and (4.2) we have $\psi_i(u) = (-1)^i\psi_i(1-u), i = 0,1,2,\ldots$ Also, if F is symmetric then we have $F^{-1}(u) = -F^{-1}(1-u)$. This gives

$a_i = \int_0^1 \left(F^{-1}(u)\right)^{\alpha} \psi_i(u)du = \int_0^1 (-1)^{\alpha+i} \left(F^{-1}(u)\right)^{\alpha} \psi_i(u)du = (-1)^{\alpha+i} a_i, i = 0,1,2,...$ This implies that $a_i = 0 \text{ if } \alpha + i \text{ is odd}.$ Also we have from (4.1),

$$a_i = \int_0^1 \left\{F^{-1}(u)\right\}^{\alpha} \psi_i(u)du \quad = \sqrt{2i+1} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \binom{i+j}{j} \int_0^1 \left(F^{-1}(u)\right)^{\alpha} u^j du$$

Putting $t = F^{-1}(u)$ we get

$$a_i = \sqrt{2i+1} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \binom{i+j}{j} \int_{-\infty}^{\infty} t^{\alpha} (F(t))^j \frac{d}{dt} F(t)$$

$$= \sqrt{2i+1} \sum_{j=0}^{i} \frac{(-1)^{i-j}}{j+1} \binom{i}{j} \binom{i+j}{j} EY^{\alpha}_{j+1,\, j+1}$$

where $Y_{j+1,\, j+1}$ is the maximum in a random sample of size j+1 drawn from F. Thus (4.3) is proved.

(b) Now from (4.2) we have

$$b_i(n,k) = \int_0^1 \frac{k^n}{(n-1)!} \{-log(1-u)\}^{n-1} (1-u)^{k-1} \psi_i(u)\, du$$

$$= \frac{k^n}{(n-1)!} \sqrt{2i+1} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{i+j}{j} \int_0^1 \{-log(1-u)\}^{n-1} (1-u)^{k+j-1} du$$

The substitution $t = -\log(1-u)$ gives,

$$b_i(n,k) = \frac{k^n}{(n-1)!} \sqrt{2i+1} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{i+j}{j} \int_0^{\infty} \mathrm{e}^{-(k+j)t}\, \mathrm{t}^{n-1} dt$$

$$= k^n \sqrt{2i+1} \sum_{j=0}^{i} \frac{(-1)^j}{(k+j)^n} \binom{i}{j} \binom{i+j}{j}.$$

This proves (4.4).

(c) From the definitions of $b_i(n_1, n_2, k)$ and $b_i(n,k)$, (4.5) is immediate.

(d) We have

$$b_i^*(n_1,n_2,k) = \int_0^1 \frac{k^{n_2}}{2(n_2-1)!} ((-1)^{\alpha} (-logu)^{n_2-1} u^{k-1} + (-\log(1-u))^{n_2-1} (1-u)^{k-1}) \psi_i(u)\, du -$$
$$\int_0^1 \frac{k^{n_1}}{2(n_1-1)!} ((-1)^{\alpha} (-logu)^{n_1-1} u^{k-1} + (-\log(1-u))^{n_1-1} (1-u)^{k-1}) \psi_i(u)\, du$$
$$= \frac{b_i(n_1,n_2,k)}{2} + (-1)^{\alpha} \int_0^1 \frac{k^{n_2}}{2(n_2-1)!} (-logu)^{n_2-1} u^{k-1} \psi_i(u)\, \mathrm{du} -$$
$$(-1)^{\alpha} \int_0^1 \frac{k^{n_1}}{2(n_1-1)!} (-logu)^{n_1-1} u^{k-1} \psi_i(u)\, \mathrm{du}$$

Putting y= 1-u and noting that $\psi_i(u) = (-1)^i \psi_i(1-u)$ and simplifying, we get (4.6).

5. Applications

In this section, we compute the bounds when F is standard normal, $\alpha = 2$ and k=1. Throughout this section, ψ denotes the standard normal distribution function and ϕ its derivative. For any integer $n(\geq 0)$, put $I_n = \int_{-\infty}^{\infty} x\psi^n(x)\phi^2(x)dx.$

Note that $I_0 = 0.$

Let $A_{n_1,n_2} = \frac{1}{n_1!n_2!}\int_0^1 \{-\log u\}^{n_1}\{-\log(1-u)\}^{n_2} du$ where n_1 *and* $n_2 (n_1 \leq n_2)$ are positive integers. Observe that $A_{n_1,n_2} = A_{n_2,n_1}$. Put $A_n = A_{n,n}$ where n is a positive integer.

Let us denote $b_i(n,1)$ by $b_i(n)$. Computer programs were written using C language for the computation of I_n, A_{n_1,n_2} a_i *and* $b_i(n)$. Note that $b_i(n)$'s are distribution free and depend only on n and i. For the computation of I_n the interval (-4,4) was divided into 1,000 subintervals. Similarly, for the computation of A_{n_1,n_2}, the interval (0,1) was subdivided into 1000 subintervals. The values for a_i, b_i *and* A_{n_1,n_2} are given in the following tables.

Table 5.1: Values of a_i for the standard normal distribution

i	a_i
0	1.000000000000
2	1.232769191294
4	0.520873554120
6	0.304288401952
8	0.205401695559
10	0.150617703658
12	0.116802538144

Note: $a_{2i+1} = 0, i = 0,1,2,3,....$

Table 5.2: Values of $b_i(n)$

i/n	2	3	4	5	6
0	1	1	1	1	1
1	0.866025404	1.299038106	1.515544457	1.623797632	1.677924220
2	0.372677996	1.055920989	1.563177151	1.872016787	2.044840456
3	0.220479276	0.790050739	1.434646400	1.928555703	2.245632887
4	0.150000000	0.617500000	1.278708333	1.893154861	2.343771308
5	0.110554160	0.501178857	1.139751967	1.820877169	2.379730886
6	0.085846459	0.418603685	1.022663948	1.737030837	2.378819160
7	0.069160417	0.357411154	0.924770928	1.652313850	2.356219291
8	0.057265356	0.310482761	0.842492535	1.571127032	2.320984259
9	0.048432210	0.273488236	0.772711284	1.495120119	2.278583606
10	0.041659779	0.243661615	0.712950030	1.424717490	2.232376460
11	0.036332057	0.219161469	0.661286883	1.359799977	2.184453921
12	0.032051282	0.198718304	0.616233962	1.300019804	2.136128185

Table 5.3: Values of A_{n_1,n_2} $(1 \leq n_1 \leq n_2 \leq 5)$.

n_1/n_2	1	2	3	4	5
1	0.355191857	0.153072864	0.070716038	0.033772402	0.016421698
2	**	0.035450466	0.009575879	0.002794231	0.000852448
3	**	**	0.001654557	0.000323925	0.000068280
4	**	**	**	0.000044704	0.000006850
5	**	**	**	**	0.000000787

Note: $A_n = A_{n,n}, n = 1,2,3,4,5.$

Table 5.4: Values of I_n.

N	1	2	3	4	5	6
I_n	0.045928072	0.045882817	0.039916322	0.033963691	0.028958200	0.024896327
N	7	8	9	10	11	12
I_n	0.021610744	0.018935008	0.016734265	0.014905091	0.013369082	0.012066854
N	13	14	15	16	17	18
I_n	0.010953031	0.009992631	0.009158364	0.008428773	0.007786760	0.007218615
N	19	20	21	22	23	24
I_n	0.006713211	0.006261455	0.005855867	0.005490233	0.005159360	0.004858885

We need the following two lemmas.

Lemma 5.1*:* Let $Y_{n,n}$ denote the maximum in a random sample of size n drawn from standard normal distribution. Then

$EY^2_{i+1,i+1} = 1 + i(i+1)I_{i-1}, i = 0,1,2,3,...$ *where* $I_{-1} = 0.$

Proof: Observe that $\frac{d}{dy}\psi(y) = -y\phi(y).$ Differentiating on both sides,

$\frac{d^2}{dy^2}\psi(y) = -y\frac{d}{dy}\phi(y) - \phi(y) = y^2\phi(y) - \phi(y)$. This gives, $y^2\varphi(y) = \varphi(y) + \frac{d^2}{dy^2}\psi(y) = \frac{d}{dy}\{\psi(y) - y\phi(y)\}.$ Hence

$$\int y^2\varphi(y)dy = \psi(y) - y\varphi(y).$$

We have $EY^2_{i+1,i+1} = (i+1)\int_{-\infty}^{\infty} y^2\psi^i(y)\phi(y)dy.$ Integrating by parts, by taking $y^2\varphi(y)$ as the second function we get

$$EY^2_{i+1,i+1} = i + 1 - i(i+1)\int_{-\infty}^{\infty}\psi^i(y)\phi(y)dy + i(i+1)\int_{-\infty}^{\infty} y\psi^{i-1}(y)\phi^2(y)dy$$

$= 1 + i(i+1)I_{i-1}, i = 0,1,2,3,...$ where we define $I_{-1} = 0$.

Lemma 5.2: Let $\{\psi_i, i \geq 0\}$ be a complete system of Legendre polynomials in $L^2(0,1)$. Let $a_i = \int_0^1\left\{\left(\psi^{-1}(u)\right)\right\}^2\psi_i(u)du$ where ψ is standard normal d.f. Then

$$a_{2i+1} = 0, i = 0,1,2,... \text{ and } a_{2i} = \sqrt{4i+1}\sum_{j=2}^{2i} j\,(-1)^j\binom{2i}{j}\binom{2i+j}{j}I_{j-1}, i = 1,2,3,....$$

Proof: Since $\alpha = 2,$ by theorem 4.1(a), it follows that $a_i = 0$ when i is odd.

Also, by (4.3) we have $a_i = \sqrt{2i+1}\sum_{j=0}^{i}\frac{(-1)^{i-j}}{j+1}\binom{i}{j}\binom{i+j}{j}EY^{\alpha}_{j+1,j+1}$, $i = 0,1,2,3,...$

where $Y_{i+1,i+1}$ is the maximum in a random sample of size $i+1$ from the standard normal d.f.

By lemma 5.1, we now have

$$a_i = \sqrt{2i+1}\sum_{j=0}^{i}\frac{(-1)^{i-j}}{j+1}\binom{i}{j}\binom{i+j}{j}\left(1 + j(j+1)I_{j-1}\right)$$

This gives

$a_i = \sqrt{2i+1} \sum_{j=2}^{i} (-1)^{i-j} \binom{i}{j}\binom{i+j}{j} jI_{j-1}$ since $\sum_{j=0}^{i} \frac{(-1)^{i-j}}{j+1}\binom{i}{j}\binom{i+j}{j} = 0$ and $I_{-1} = I_0 = 0.$

We now obtain the bounds for the differences of moments of record values from the standard normal distribution through the following theorem. We denote X(n,1) by X(n).

Theorem 5.1: Let $n_1, n_2 \ (0 \prec n_1 \prec n_2)$ and $m \geq 0$ be integers. Then we have,

$$R_{m,n_1,n_2} - B_{m,n_1,n_2} \prec EX^2(n_2) - EX^2(n_1) \prec R_{m,n_1,n_2} + B_{m,n_1,n_2} \qquad (5.1)$$

where

$$R_{m,n_1,n_2} = \sum_{i=0}^{m} a_i b_i^*,$$

$$B_{m,n_1,n_2} = \left\{3 - \sum_{i=0}^{m} a_i^2\right\}^{1/2} \left\{ \frac{\binom{2n_1-2}{n_1-1}}{2} + \frac{\binom{2n_2-2}{n_2-1}}{2} + \frac{A_{n_1}}{2} + \frac{A_{n_2}}{2} - \binom{n_1+n_2-2}{n_1-1} - A_{n_1,n_2} - \sum_{i=0}^{m} b_i^{*2} \right\}^{1/2},$$

$$a_{2i+1} = 0, \ a_{2i} = \sqrt{4i+1} \sum_{j=2}^{2i} j\,(-1)^j \binom{2i}{j}\binom{2i+j}{j} I_{j-1},$$

$$b_{2i+1}^* = 0 \ \ and \ \ b_{2i}^*(n_1,n_2) = b_{2i}(n_2) - b_{2i}(n_1), i = 0,1,2,3,.....$$

Proof: Follows from theorem 3.2, theorem 4.1 and lemma 5.2.

The above bounds have been calculated and tabulated below.

Table 5.5: Bounds for $EX^2(2) - EX^2(1)$ in the case of standard normal distribution

m	**Bound 5.1**
0	0.0000 $\pm$ 1.4438
2	0.8423 $\pm$ 0.5257
4	1.0858 $\pm$ 0.2731
6	1.1870 $\pm$ 0.1693
8	1.2391 $\pm$ 0.1162
10	1.2695 $\pm$ 0.0853
12	1.2889 $\pm$ 0.0655

Table 5.6: Bounds for $EX^2(3) - EX^2(2)$ in the case of standard normal distribution

m	**Bound (5.1)**
0	0.0000 $\pm$ 2.4532
2	0.6253 $\pm$ 1.1496
4	0.9697 $\pm$ 0.6955
6	1.1535 $\pm$ 0.4763
8	1.2628 $\pm$ 0.3516
10	1.3335 $\pm$ 0.2729
12	1.3823 $\pm$ 0.2195

Table 5.7: Bounds for $EX^2(4) - EX^2(3)$ in the case of standard normal distribution

m	**Bound 5.1**
0	0.0000 $\pm$ 4.4723
2	0.3807 $\pm$ 2.1811
4	0.7008 $\pm$ 1.4110
6	0.9182 $\pm$ 1.0244
8	1.0678 $\pm$ 0.7935
10	1.1750 $\pm$ 0.6411
12	1.2549 $\pm$ 0.5333

Table 5.8: Bounds for $EX^2(5) - EX^2(4)$ in the case of standard normal distribution

m	**Bound 5.1**
0	0.0000 $\pm$ 8.3666
2	0.2131 $\pm$ 4.0982
4	0.4478 $\pm$ 2.6954
6	0.6431 $\pm$ 1.9996
8	0.7971 $\pm$ 1.5834
10	0.9187 $\pm$ 1.3065
12	1.0164 $\pm$ 1.1083

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