Characterizations of Distributions using Moments of k-Records

S. S. Nayak[1*], Bharati Gururajrao[2]

[1,2] Department of Statistics, Gulbarga University, Gulbarga-585106, Karnataka, INDIA.

[*]Corresponding Address:

ssnayak2006@rediffmail.com

Research Article

Abstract: In this paper, we characterize some distributions using the moments of k-records associated with a sequence of i.i.d continuous random variables. The k-records are special cases of the (upper) generalized order statistics introduced by Kamps (1995). The basic idea comes from the Cauchy-Schwarz inequality.

Key Words: Generalized order statistics, k-record values, Cauchy-Schwarz inequality, Weibull distribution.

2010 Mathematics Subject Classification: 62G30,62E10.

1. Introduction

The concept of generalized order statistics (gos) was introduced by Kamps (1995). The random variables U(r,n,$\tilde{m}$,k), 1≤ r≤ n are called uniform upper generalized order statistics if they possess a joint probability density function of the form

$g(u_1, u_2, \dots \dots . u_n) = k\left(\prod_{j=1}^{n-1} \gamma_j\right) \left(\prod_{i=1}^{n-1} (1-u_i)^{m_i}\right) (1-u_n)^{k-1}$on the cone $0 \le u_1 \le u_2 \le \cdots \dots \dots . . \le u_n < 1$ of the n-dimensional Euclidean space R^n , where n≥2, k>0 and $\tilde{m} = (m_1, m_2, \dots \dots m_{n-1}) \in R^{n-1}$ are parameters such that $\gamma_r = k + n + r + \sum_{j=r}^{n-1} m_j > 0$ for all $r \in \{1,2, \dots . . n-1\}$ $(\gamma_n = k)$. Consider a sequence $\{X_n, n \ge 1\}$ of independent and identically distributed (iid) random variables (r,v's) with a common continuous distribution function (df) F. The upper generalized order statistics (ugos) based on F (or ugos associated with $\{X_n, n \ge 1\}$) are defined by the quantile transformation X(r,n,$\tilde{m}$, k) = $F^{-1}(U(r, n, \tilde{m}, k)), 1 \le r \le n$. When $m_1 = m_2 = \cdots . . = m_{n-1} = -1$, the resulting ugos are called upper k-record values. We denote the nth upper k-record value by X(n,k). A fairly good account of gos and related statistics can be found in Kamps (1995). For characterizations using gos see Ahsanullah, M. (2000,2005), Claudia Keseling (1999), Claudia Keseling and Kamps, U.(2003), Cramer,E., Kamps,U. and Raqab, M. (2000), Kamps,U. and Gather, U. (1997), Nayak, S. S. and Kunichi, M. C. (2006,2008a,2008b), Raqab, M.Z. and Lina N. Abu-Lawi (2004) and Khan et.al.(2013). In this paper, we characterize the shifted Weibull distribution and some others using the moments of the ugos.

2. Preliminaries

Lemma 2.1: If $E|X_1|^\gamma < \infty$ then $E|X(n,k)|^\delta < \infty \; for \; all \; \delta < \gamma$ and all n ≥ 1 where δ and γ are positive integers.

Proof: Let U(n,k) be the nth k -record associated with the standard uniform distribution. Then X(n,k) = F^{-1}(U(n,k)). The p.d.f of U(n,k) is (Kamps, 1995)

$\frac{k^n}{(n-1)!} \{-\log(1-x)\}^{n-1} (1-x)^{k-1}, 0 < x < 1.$

Hence $E|X(n,k)|^\delta = \int_0^1 \frac{k^n}{(n-1)!} |F^{-1}(x)|^\delta \{-\log(1-x)\}^{n-1} (1-x)^{k-1} dx = \frac{k^n}{(n-1)!} \int_0^1 (|F^{-1}(x)|^\gamma)^{\frac{\delta}{\gamma}} (\{-\log(1-x)\}^{\frac{(n-1)\gamma}{\gamma-\delta}} (1-x)^{\frac{(k-1)\gamma}{\gamma-\delta}})^{1-\frac{\delta}{\gamma}} dx \le \frac{k^n}{(n-1)!} (\int_0^1 |F^{-1}(x)|^\gamma dx)^{\frac{\delta}{\gamma}} (\int_0^1 \{-\log(1-x)\}^{\frac{(n-1)\gamma}{\gamma-\delta}} (1-x)^{\frac{(k-1)\gamma}{\gamma-\delta}} dx)^{1-\frac{\delta}{\gamma}}$

By Holder's inequality. The right side is finite since $\int_0^1 |F^{-1}(x)|^\gamma dx = E\,|X|^\gamma < \infty$

by hypothesis and

$$\int_0^1 \{-\log(1-x)\}^{\frac{(n-1)\gamma}{\gamma-\delta}} (1-x)^{\frac{(k-1)\gamma}{\gamma-\delta}} dx = \frac{\Gamma\left(\frac{(n-1)\gamma}{\gamma-\delta}+1\right)}{\left(\frac{k\gamma-\delta}{\gamma-\delta}\right)^{\frac{(n-1)\gamma}{\gamma-\delta}+1}} \; is \; finite.$$

Lemma 2.2: Let f and g be any two square integrable continuous functions defined on (0,1). Let $a = \int_0^1 f(x)dx \; and \; b = \int_0^1 g(x)dx$. Then

$$|\int_0^1 f(x)g(x)dx - ab| = (\int_0^1 f^2(x)dx - a^2)^{\frac{1}{2}} (\int_0^1 g^2(x)dx - b^2)^{\frac{1}{2}} \quad (2.1)$$

if and only if

$f(x) - a = \lambda(g(x) - b)$ for some real λ **.**

Proof: We have

$$\left|\int_0^1 f(x)g(x)dx - ab\right| = \left|\int_0^1 (f(x)-a)(g(x)-b)dx\right|$$

$\le (\int_0^1 (f(x)-a)^2 dx)^{\frac{1}{2}} (\int_0^1 (g(x)-b)^2 \, dx)^{\frac{1}{2}}$ by Cauchy-Schwarz inequality.

$= (\int_0^1 f^2(x)dx - a^2)^{\frac{1}{2}} (\int_0^1 g^2(x)dx - b^2)^{\frac{1}{2}}.$

Equality holds in this inequality if and only if $f(x) - a = \lambda(g(x) - b)$ for some real λ.

3. Characterizations

In this section, $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed (iid) random variables (r.v's) with a common continuous distribution function (df) F. We assume that $E|X_1|^{2\alpha} < \infty$ so that $E|X_1|^{\alpha} < \infty$ where α is a positive integer. $\{X(n,k), n \geq 1\}$ is the corresponding sequence of nth k-record values. By lemma 2.1, it follows that $E|X(n,k)|^{\alpha} < \infty$. Put $\mu_r' = EX_1^r$, $0<r\leq 2\alpha$.

Note that $EX^{\alpha}(n,k) = \int_0^1 \frac{k^n}{(n-1)!} U(u)^{\alpha}\{-\log(1-u)\}^{n-1}(1-u)^{k-1}du$ (3.1)

$where\ U(u) = F^{-1}(u)$.

Theorem 3.1:

a) If either k>1, α ≥ 1, n ≥ 2 or k=1, α ≥ 2(even) , n ≥ 2 and $F^{-1}(0) < 0$ then there does not exist any d.f. F such that

$$|E\,X^{\alpha}(n,k) - \mu_{\alpha}'| = (\mu_{2\alpha}' - \mu_{\alpha}'^{\,2})^{1/2}\left\{\binom{2n-2}{n-1}\frac{k^{2n}}{(2k-1)^{2n-1}} - 1\right\}^{1/2} \quad (3.2)$$

b) Let k=1, α ≥ 1, n ≥ 2 and $F^{-1}(0) \geq 0$. Then (3.2) holds if and only if

$$F(t) = 1 - exp\left(-\left(\frac{(n-1)!(\lambda-\mu_{\alpha}'+t^{\alpha})}{\lambda}\right)^{\frac{1}{n-1}}\right), t > (\mu_{\alpha}' - \lambda)^{\frac{1}{\alpha}}, \lambda > 0.$$

c) Let k=1, α (≥ 1) odd , n ≥ 2 and $F^{-1}(0) < 0$. Then (3.2) holds if and only if

$$F(t) = 1 - exp\left(-\left(\frac{(n-1)!(\lambda-\mu_{\alpha}'-(-t)^{\alpha})}{\lambda}\right)^{\frac{1}{n-1}}\right), \text{ if } -(\lambda - \mu_{\alpha}')^{\frac{1}{\alpha}} < t \leq 0 \text{ and}$$

$$F(t) = 1 - exp\left(-\left(\frac{(n-1)!(\lambda-\mu_{\alpha}'+t^{\alpha})}{\lambda}\right)^{\frac{1}{n-1}}\right), if\ 0 \leq t < \infty\ where\ \lambda > 0.$$

Proof: Taking $f(u) = \{F^{-1}(u)\}^{\alpha}$ and $g(u) = \frac{k^n}{(n-1)!}\{-\log(1-u)\}^{n-1}(1-u)^{k-1}$

In (2.1) we notice that $\int_0^1 f(x)g(x)dx = EX^{\alpha}(n,k)$ (from (3.1)) , a = μ_{α}' , b =1,

$\int_0^1 f^2(x)dx = \mu_{2\alpha}'$ and

$\int_0^1 g^2(x)dx = \int_0^1 \frac{k^{2n}}{((n-1)!)^2}\{-\log(1-u)\}^{2(n-1)}(1-u)^{2(k-1)}du.$

The substitution $-\log(1-u) = t$ gives

$$\int_0^1 g^2(u)du = \int_0^{\infty} \frac{k^{2n}}{((n-1)!)^2}\, e^{-(2k-1)t}\, t^{2(n-1)}\, dt = \binom{2n-2}{n-1}\frac{k^{2n}}{(2k-1)^{2n-1}}.$$

Now (2.1) reduces to (3.2).

By (2.2), (3.2) holds if and only if $\{F^{-1}(u)\}^{\alpha} = \mu_{\alpha}' + \lambda(g(u)-1)$ for some λ real. (3.3)

a) First let k >1, α ≥ 1 and n ≥ 2. Since k>1 , g(u) is increasing in theinterval (0, u_0) and decreasing in the interval (u_0, 1) where ,

$u_0 = 1 - e^{\frac{n-1}{k-1}}$. Hence the right side of (3.3) is not monotonic in (0,1). But, the left side of (3.3) is non-increasing in (0,1). Hence (3.3) cannot be satisfied for any d.f F.

Now let k=1, α(≥2) even ,n ≥ 2 and $F^{-1}(0) < 0$. Note that $g(u) = \frac{(-\log(1-u))^{n-1}}{(n-1)!}$,

and $g'(u) > 0$, 0 < u < 1. Since the left side of (3.3) is non-decreasing, we must have λ > 0. Hence the right side of (3.3) is negative for

$$0 < u < u_1 = 1 - exp\left(-\left(\frac{(\lambda - \mu_{\alpha}')(n-1)!}{\lambda}\right)^{\frac{1}{n-1}}\right).$$ But, the left side of (3.3) is positive since α (≥2) is even. Thus (3.3) cannot be satisfied for any d.f F.

b) Since k = 1, we have $g'(u) > 0$, 0 < u < 1. Since the left side of (3.3) is non-decreasing , we must have λ > 0. Since $F^{-1}(0) \geq 0$ and g(0) = 0, it follows that $\mu_{\alpha}' - \lambda \geq 0$ and $F^{-1}(u) = \{\mu_{\alpha}' + \lambda(g(u)-1)\}^{\frac{1}{\alpha}}$, 0 < u < 1.

Note that $g'(x) = 1 - exp\left\{-(x(n-1)!)^{\frac{1}{n-1}}\right\}$, x > 0.

Putting $t = \{\mu_{\alpha}' + \lambda(g(u)-1)\}^{\frac{1}{\alpha}}$, we notice that $t > (\mu_{\alpha}' - \lambda)^{\frac{1}{\alpha}} \geq 0$ and

$F(t) = g^{-1}\left(\frac{t^{\alpha}+\lambda-\mu_{\alpha}'}{\lambda}\right)$

$$= 1 - \exp\left\{-\left(\frac{(t^{\alpha} + \lambda - \mu_{\alpha}^{'})(n-1)!}{\lambda}\right)^{\frac{1}{n-1}}\right\}, t > (\mu_{\alpha}^{'} - \lambda)^{\frac{1}{\alpha}}, \lambda > 0$$

c) Let k = 1, α (≥ 1) odd, n ≥ 2 and $F^{-1}(0) < 0$. As in case b) , we have λ > 0.
Since $F^{-1}(0) < 0$ and α is odd , we observe from (3.3) that $\mu_{\alpha}^{'} - \lambda < 0$,
$F^{-1}(u) < 0$ for $0 < u < u_1$ and $F^{-1}(u) > 0$ for $u_1 < u < 1$, where u_1 is as in case a).
Let $0 < u < u_1$. Then $F^{-1}(u) = -h(u)$, where $h(u) > 0$. From (3.3) , we have $(-\square(u))^{\alpha} = \mu_{\alpha}^{'} + \lambda\,(g(u) - 1)$. This gives

$$F^{-1}(u) = -\{-(\mu_{\alpha}^{'} + \lambda\,(g(u) - 1))\}^{\frac{1}{\alpha}},\ 0 < u < u_1.$$

Proceeding as in case b) , we get

$$F(t) = 1 - exp\left\{-\left(\frac{(n-1)!(\lambda - \mu_{\alpha}^{'} - (-t)^{\alpha}}{\lambda}\right)^{\frac{1}{n-1}}\right\}, -(\lambda - \mu_{\alpha}^{'})^{\frac{1}{\alpha}} < t < 0\,.$$

Now let $u_1 < u < 1$. In this case $F^{-1}(u) = \{\mu_{\alpha}^{'} + \lambda(g(u) - 1)\}^{\frac{1}{\alpha}}$, $u_1 < u < 1$.
Proceeding as in case b) , we get

$$F(t) = 1 - exp\left\{-\left(\frac{(n-1)!(\lambda - \mu_{\alpha}^{'} + t^{\alpha}}{\lambda}\right)^{\frac{1}{n-1}}\right\}, t > 0.$$

Remark: 3.1: In case b) , the d.f is Weibull distribution with shifted origin.

Theorem 3.2: Let F be a continuous d.f symmetric about the origin. Let
k , α and n be integers such that

(i) k > 1, α ≥ 1, n ≥ 2 or
(ii) k = 1, α ≥ 2(even), n ≥ 2 or
(iii) k = 1, α (≥ 1) odd, n = 2.

a) In cases (i) and (ii) there does not exist any d.f F such that

$$\left|EX^{\propto}(n,k) - b\mu_{\alpha}^{'}\right| = (\mu_{2\alpha}^{'} - \mu_{\alpha}^{'\,2})^{\frac{1}{2}}\left(\frac{k^{2n}}{2}\left(\frac{\binom{2n-2}{n-1}}{(2k-1)^{2n-1}} + (-1)^{\alpha}A_{n_k}\right) - b^2\right)^{\frac{1}{2}} \tag{3.4}$$

where $b = \frac{1+(-1)^{\alpha}}{2}$ and $A_{n,k} = \frac{1}{((n-1)!)^2}\int_0^1 (\log(1-u))^{n-1}(1-u)^{k-1}(\log u)^{n-1}\,u^{k-1}du$.

b) In case (iii), (3.4) is true if and only if

$$F(t) = \frac{1}{1+\exp\left(\frac{-2t^{\alpha}}{\lambda}\right)},\ -\infty < t < \infty, \lambda > 0.$$

Proof: Take $f(u) = \{F^{-1}(u)\}^{\alpha}$ and

$$g(u) = \frac{k^n}{2(n-1)!}((-\log(1-u))^{n-1}(1-u)^{k-1} + (-1)^{\alpha}(-\log u)^{n-1}\,u^{k-1})$$

In (2.1). Then $\int_0^1 f(u)g(u)du = \frac{k^n}{2(n-1)!}\int_0^1 U^{\alpha}(u)(-\log(1-u))^{n-1}(1-u)^{k-1}du$
$+\frac{k^n(-1)^{\alpha}}{2(n-1)!}\int_0^1 U^{\alpha}(u)(-\log u)^{n-1}\,u^{k-1}du$.
$= \frac{1}{2}EX^{\alpha}(n,k) + \frac{k^n(-1)^{\alpha}}{2(n-1)!}\int_0^1\{-U(1-u)\}^{\alpha}(-logu)^{n-1}u^{k-1}du.$
(from (3.1) and the fact that the symmetry of F about zero implies
$U(u) = -\square(1-\square), 0 < \square < 1.$)
$= \frac{1}{2}\square\square^{\square}(\square,\square) + \frac{\square^{\square}}{2(\square-1)!}\int_0^1 \square^{\square}(\square)(-\log(1-\square))^{\square-1}(1-\square)^{\square}\square\square$
$= \square\square^{\square}(\square,\square)$.
Also, $a = \square_{\square}^{'}$, $b = \frac{1+(-1)^{\square}}{2}$, $\int_0^1 \square^2(\square)\square\square = \square_{2\square}^{'}$ and
$\int_0^1 \square^2(\square)\square\square = \frac{\square^{2\square}}{2}\left(\frac{\binom{2\square-2}{\square-1}}{(2\square-1)^{2\square-1}} + (-1)^{\square}\square_{\square,\square}\right)$, where $A_{n,k}$ is as in (a).
Then (2.1) reduces to (3.4). By (2.2), (3.4) holds good if and only if

$$\{\square^{-1}(\square)\}^{\square} - \square_{\square}^{'} = \square(\square(\square) - \square) \text{ for some } \lambda > 0. \tag{3.5}$$

a) Note that the left side of (3.5) is non-decreasing in (0,1) whereas then right side is not non-decreasing in (0,1) in cases (i) and (ii). Hence there does not exist any d.f F for which (3.4) is true.
b) Let k = 1, n = 2 and α ≥ 1 (odd). Then (3.5) reduces to

$$\{\square^{-1}(\square)\}^{\square} = \frac{\square}{2}\log\left(\frac{\square}{1-\square}\right),\ 0 < u < 1.$$

Since $(F^{-1}(u))^{\alpha}$ is non-decreasing , it follows that $\lambda > 0$. Since α is odd, we see that $F^{-1}(u)$ is negative or positive according as $o < u < ½$ or $½ < u < 1$.
First let $0 < u < ½$. Then

$$\square^{-1}(\square) = -\left(\frac{\square}{2}\right)^{\frac{1}{\square}}\left(\square\square\square\left(\frac{1-\square}{\square}\right)\right)^{\frac{1}{\square}}.$$

Putting $\square = -\left(\frac{\square}{2}\right)^{\frac{1}{\square}}\left(\square\square\square\left(\frac{1-\square}{\square}\right)\right)^{\frac{1}{\square}}$ and solving for u, we get

$$\square(\square) = \frac{1}{1+\square^{\frac{2}{\square}(-\square)^{\square}}}, \square < 0.$$

Similarly for $½ < u < 1$, we get

$$\square(\square) = \frac{1}{1+\square^{\frac{-2}{\square}\square^{\square}}}., \square > 0.$$

Since α is odd, we observe that

$$\square(\square) = \frac{1}{1+\square^{\frac{-2}{\square}\square^{\square}}}, -\infty < \square < \infty, \square > 0.$$

Remark 3.2: Note that in case b) F reduces to the logistic d.f when $\alpha = 1$.

Acknowledgement: The first author is grateful to The Dept. of Science and Technology (DST), New Delhi for financial assistance. (No. SR/S4/MS:644/10 dated 21-2-11).

References

1. Kamps, U. (1995): A Concept of Generalized Order Statistics. Journal of Statistical Planning and Inference, 48, pp. 1-23.
2. Ahsanullah, M. (2000): Generalized Order Statistics from Exponential Distribution. Journal of Statistical Planning and Inference. 85, pp. 85-91.
3. Ahsanullah, M. (2005): On Lower Generalized Order Statistics and a Characterization of the Power Function Distribution. Statistical Methods, 7(1), pp. 16-28.
4. Claudia Keseling (1999): Conditional Distributions of Generalized Order Statistics and some Characterizations. Metrika, Vol. 49, pp. 27-40.
5. Claudia Keseling and Kamps, U. (2003): A Theorem of Rossberg of Generalized Order Statistics. Sankhya, Vol. 65, Pt.2, pp. 259-270.
6. Cramer,E., Kamps, U. and Raqab, M. (2000): Characterizations of Exponential Distributions by Spacings of Generalized Order Statistics.
7. Kamps, U. and Gather, U. (1997): Characteristic Properties of Generalized Order Statistics from Exponential Distributions. Applications Mathematicae, 24, 4, pp. 383-391.
8. Nayak, S.S. and Kunichi, M.C. (2006): Characterizations Using Independence of Functions of Generalized Order Statistics. JISA, VOl. 44, No. 2, pp. 137-155.
9. Nayak, S.S. and Kunichi, M.C. (2008a): Characterizations by Regression of Generalized Order Statistics. IJASS, Vol. 4, No. 2, pp. 365-378.
10. Nayak, S.S. and Kunichi, M. C. (2008b): Characterizations Using Equidistribution of Functions of Generalized Order Statistics. JISA, Vol. 46, No. 1, pp. 43-53.
11. Raqab, M.Z. and Lina N. Abu-Lawi (2004): Characterizations of Continuous Distributions based on expectations of Generalized Order Statistics. Vol. 13, No. 2, pp. 101-116.