Bayesian Estimation of Three-Parameter Exponentiated Log-logistic Distribution

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Research Article

Abstract: In this paper, we have considered the estimation problem of three-parameter exponentiated log-logistic distribution. The parameters are estimated using likelihood based inferential procedure: classical as well as Bayesian. We have computed MLEs and Bayes estimates under informative and non-informative priors along with their asymptotic confidence, bootstrap and HPD intervals. The Bayesian estimates of the parameters of exponentiated log-logistic distribution are obtained using Markov chain Monte Carlo (MCMC) simulation method. We have obtained the probability intervals for parameters, hazard and reliability functions. The posterior predictive check procedure has been applied for evaluating the model fit. All the computations are performed in OpenBUGS and R software. A real data set is analyzed for illustration of the proposed inferential procedures.

Keywords: Exponentiated log-logistic distribution, Markov chain Monte Carlo, Bayesian estimation, Bootstrap, OpenBUGS.

1. Introduction

There are hundreds of continuous univariate distributions available in literature. Numerous classical distributions have been extensively used over the past decades for modeling data in several areas such as engineering, actuarial, environmental and medical sciences, biological studies, demography, economics, finance, and insurance. However, in many applied areas like lifetime analysis, finance, and insurance, there is a clear need for extended forms of these distributions, that is, new distributions which are more flexible to model real data in these areas, since the data can present a high degree of skewness and kurtosis. So, we can give additional control over both skewness and kurtosis by adding new parameters, and hence, the extended distributions become more flexible to model real data. Recent developments focus on new techniques for building meaningful distributions.

The modeling and analysis of lifetimes is an important aspect of statistical work in a wide variety of scientific and technological fields. The properties of the log-logistic distribution make it an attractive alternative to the log normal and Weibull distributions in the analysis of survival data, Bennett (1983) and Tadikamalla and Johnson (1982). It has a nonmonotonic hazard function. The shape of this distribution is very similar to that of the log-normal, but has a more tractable form than that of the log-normal which makes it more convenient than the log-normal distribution when dealing with censored data. Srivastava and Shukla (2008) studied the log-logistic distribution as step-stress model. Balakrishnan and Malik (1987) gave the moments of order statistics from the truncated log-logistic distribution. This distribution has been also studied by Howlader and Weiss (1992). Lawless (2003) and Lee and Wang(2003) provide an excellent review for the log-logistic distribution.

In recent years, new classes of models have been proposed based on modifications such as adding parameters to the existing models. Adding one or more parameters to a distribution makes it richer and more flexible for modeling data.

There are different ways for adding parameter(s) to a distribution. Marshall and Olkin (1997, 2007) added one positive parameter to a given (general) survival function. As described by Marshall and Olkin (2007) and Klugman et al.(2012), an exponentiated distribution can be easily constructed. It is based on the observation that by raising any baseline cumulative distribution function (cdf) $F_{baseline}(x)$ to an arbitrary power $\alpha > 0$, a new cdf

$$F(x) = \left(F_{baseline}(x)\right)^{\alpha} \; ; \alpha > 0$$

is obtained with the additional parameter α. Following this idea, several authors have considered extensions from usual survival distributions. For instance, Mudholkar and Srivastava (1993) considered the exponentiated Weibull distribution as a generalization of the Weibull distribution. Gupta and Kundu (1999) introduced the exponentiated exponential distribution as a generalization of the usual exponential distribution and Nadarajah and Kotz (2006) proposed exponentiated type distributions extending the Frchet, gamma, Gumbel and Weibull distributions. Rosaiah, et al. (2006, 2007) studied the some properties and reliability test plan for exponentiated log-logistic distribution. Chaudhary and Kumar (2013a)

obtained the maximum likelihood and Bayes estimators for the parameters and the reliability function of the two-parameter exponentiated log-logistic distribution using Markov chain Monte Carlo simulation method. Santana et al.(2012) introduced the Kumaraswamy-log-logistic distribution, which includes exponentiated log-logistic distribution.

The cumulative distribution function(cdf) of the log-logistic distribution is given by

$$F_{LL}(x;\ \beta,\ \lambda) = \frac{(x/\lambda)^{\beta}}{1+(x/\lambda)^{\beta}}\ ; x > 0$$

where $\beta > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively.

The cdf of the exponentiated log-logistic(ELL) distribution is defined by raising $F_{LL}(x)$ to the power of $\alpha > 0$, namely $F(x) = \left(F_{LL}(x)\right)^{\alpha}$. The distribution function of ELL distribution with three parameters is given by

$$F(x;\alpha,\beta,\lambda) = \left\{\frac{(x/\lambda)^{\beta}}{1+(x/\lambda)^{\beta}}\right\}^{\alpha}\ ; x > 0 \qquad (1)$$

where $\alpha > 0$ and $\beta > 0$ are shape parameters and $\lambda > 0$ is a scale parameter. The exponentiated log-logistic distribution will be denoted by $ELL(\alpha,\beta,\lambda)$. For $\alpha = 1$, the model reduces to the two-parameter log-logistic distribution. The exponentiated log-logistic model is much more flexible than the log-logistic distribution and can be used effectively for modeling lifetime data. It is to be mentioned here that this distribution has not been considered under Bayesian setup in the earlier literature.

The advent of Markov chain Monte Carlo(MCMC) sampling has flourished Bayesian inference throughout the world, wide array of disciplines. The freely available software package known as Bayesian inference using Gibbs sampling(BUGS) has been in the forefront of this proliferation since the mid-1990s. However, more recent advances in this software, leading first to WinBUGS and now to an open-source version OpenBUGS, Thomas et al. (2006), Thomas (2010) and Lunn et al. (2013), including interfaces to the open-source statistical package R, (R Development Core Team, 2013) and Albert (2009), have brought MCMC to a wider audience. Problems, which would have been intractable a decade ago, can now be solved in short order with these software packages. We will use this software in our present study.

For Bayesian estimation, we also need to assume a prior distribution for the model parameters involved in the analysis. In this paper, Bayesian analysis has been preformed under different loss function assuming independent priors for the parameters.

A major difficulty towards the implementation of Bayesian procedure is that of obtaining the posterior distribution. The process often requires the integration, which is very difficult to calculate not only for high-dimensional complex models even if dealing with low-dimensional models. In such a situation, Monte Carlo Markov chain (MCMC) methods are very useful to simulate the deviates from the posterior density and produce the good approximate results.

The rest of the paper is organized as follows. In Section 2, we have discussed the exponentiated log-logistic distribution and its properties. The point estimation procedures for the parameters of the considered model under classical set-up and the confidence/bootstrap intervals have been constructed in Section 3. In Section 4, we have developed the Bayesian estimation procedure under independent priors for the parameters, gamma priors for shape parameters and uniform prior for scale parameter. To check the applicability of the proposed methodologies, a real data set has been analysed in Section 5. In this section, the ML estimators of the parameters, approximate confidence intervals are presented. We cover Bayesian analysis using the MCMC simulation in Section 6. In this section, the Bayes estimates and credible intervals of parameters, hazard and reliability functions are presented. In Section 7 we have applied the predictive check method in order to give an assessment of the performance of the model for the given data. Finally, the conclusions have been given in Section 8.

2. The model analysis

The probability density function(pdf), corresponding to (1), is given by

$$f(x;\alpha,\beta,\lambda) = \frac{\alpha\beta(x/\lambda)^{\alpha\beta}}{x\left[1+(x/\lambda)^{\beta}\right]^{\alpha+1}}\ ; x > 0 \qquad (2)$$

The reliability/survival function is

$$R(x) = 1 - \left\{\frac{(x/\lambda)^{\beta}}{1+(x/\lambda)^{\beta}}\right\}^{\alpha}\ ;\ x > 0 \qquad (3)$$

The hazard rate function is

$$h(x) = \frac{f(x;\alpha,\ \beta,\ \lambda)}{1 - F(x;\alpha,\ \beta,\ \lambda)}\ ;\ x > 0. \qquad (4)$$

The mode is given by

$$\text{Mode} = \lambda\left(\frac{\alpha\beta-1}{\beta+1}\right)^{1/\beta}\ ;\ \alpha\beta > 1. \qquad (5)$$

The quantile function is given by

$$x_p = \lambda \left(p^{-1/\alpha} - 1 \right)^{-1/\beta} \quad ; 0 < p < 1 \qquad (6)$$

The random deviate can be generated from $ELL(\alpha, \beta, \lambda)$ by

$$x = \lambda \left(u^{-1/\alpha} - 1 \right)^{-1/\beta} \qquad (7)$$
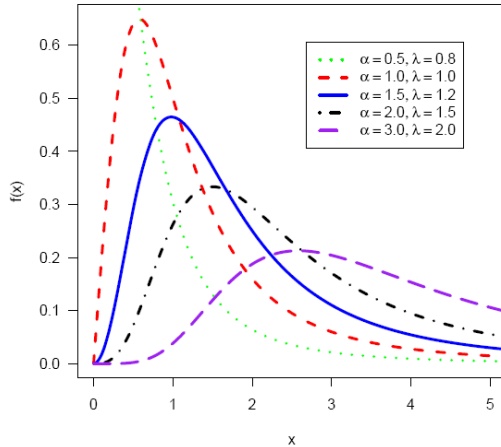
where u has the $U(0,1)$ distribution.

Figure 1: The probability density functions of ELL distribution for $\beta = 2$ and different values of α and λ.

Some of the typical exponentiated log-logistic density functions for $\beta = 2$ and different values of α and λ are depicted in Figure 1.
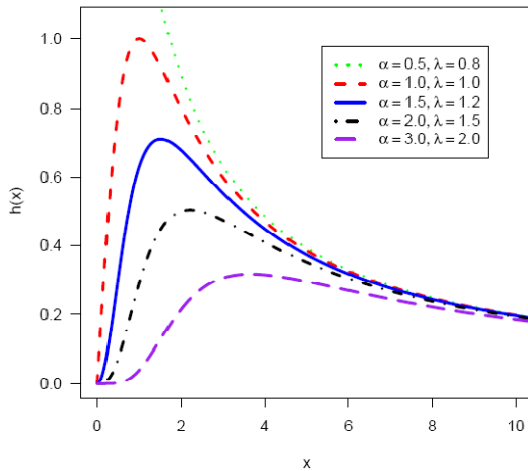

Figure 2: The hazard functions of exponentiated log-logistic distribution for $\beta = 2$ and different values of α and λ.

It is clear from the Figure 1 that the density function of the exponentiated log-logistic distribution can take different shapes. The distinct types of hazard shapes are illustrated in Figure 2 for some different parameter combinations of the exponentiated log-logistic distribution.

3. Maximum likelihood estimation (MLE)

Maximum likelihood estimation is one of the most popular methods for estimating the parameters of continuous distributions because of its attractive properties, such as consistency, asymptotic unbiased, asymptotic efficiency, and asymptotic normality. In this section, we discuss the maximum likelihood estimators (MLE's) of the $ELL(\alpha, \beta, \lambda)$ distribution and discuss their asymptotic properties to obtain approximate confidence intervals based on MLE's.

Let $\underline{x} = (x_1, \ldots, x_n)$ be a random sample of size n from $ELL(\alpha, \beta, \lambda)$, then the log-likelihood function $\ell(\alpha, \beta, \lambda \mid \underline{x})$ can be written as;

$$\ell(\alpha, \beta, \lambda \mid \underline{x}) = n \log \alpha + n \log \beta + \alpha \sum_{i=1}^{n} \log \left(\frac{x_i}{\lambda} \right)^{\beta}$$
$$- \sum_{i=1}^{n} \log x_i - (\alpha + 1) \sum_{i=1}^{n} \log \left(1 + \left(\frac{x_i}{\lambda} \right)^{\beta} \right) \qquad (8)$$

To obtain the MLE's of α, β and λ, we can maximize (8) directly with respect to α, β and λ or we can solve the following system of non-linear equations.

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log (x_i / \lambda)^{\beta} - \sum_{i=1}^{n} \log \left(1 + (x_i / \lambda)^{\beta} \right) = 0 \quad ,$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \alpha \sum_{i=1}^{n} \log (x_i / \lambda)$$
$$- (\alpha + 1) \sum_{i=1}^{n} \frac{(x_i / \lambda)^{\beta} \log (x_i / \lambda)}{\left(1 + (x_i / \lambda)^{\beta} \right)} = 0 \qquad (9)$$

$$\frac{\partial \ell}{\partial \lambda} = -\frac{n \alpha \beta}{\lambda} + \frac{\beta (\alpha + 1)}{\lambda} \sum_{i=1}^{n} \frac{(x_i / \lambda)^{\beta}}{x_i \left(1 + (x_i / \lambda)^{\beta} \right)} = 0$$

Note that the MLEs $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ of α, β and λ, respectively cannot be solved analytically. Numerical iteration techniques, such as the Newton-Raphson algorithm, are thus adopted to solve these equations.

3.1 Approximate confidence intervals

The exact distribution of MLEs cannot be obtained explicitly. Therefore, the asymptotic properties of MLEs can be used to construct the confidence intervals for the parameters. Under some regularity conditions, the MLEs $\hat{\underline{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ are approximately tri-variate normal

$$\left(\hat{\underline{\theta}} - \underline{\theta} \right) \rightarrow N_3 \left(0, (I(\underline{\theta}))^{-1} \right)$$

where $I(\underline{\theta})$ is the variance matrix. As $I(\underline{\theta})$ involves the unknown parameters, we replace these parameters by their corresponding MLEs to obtain an estimate $I(\hat{\underline{\theta}})$ as

$$I(\hat{\underline{\theta}}) = - \begin{pmatrix} \dfrac{\partial^2 \ell}{\partial \alpha^2} & \dfrac{\partial^2 \ell}{\partial \alpha\, \partial \beta} & \dfrac{\partial^2 \ell}{\partial \alpha\, \partial \lambda} \\[2mm] \dfrac{\partial^2 \ell}{\partial \beta\, \partial \alpha} & \dfrac{\partial^2 \ell}{\partial \beta^2} & \dfrac{\partial^2 \ell}{\partial \beta\, \partial \lambda} \\[2mm] \dfrac{\partial^2 \ell}{\partial \lambda\, \partial \alpha} & \dfrac{\partial^2 \ell}{\partial \lambda \partial \beta} & \dfrac{\partial^2 \ell}{\partial \lambda^2} \end{pmatrix}_{(\alpha=\hat{\alpha},\beta=\hat{\beta},\lambda=\hat{\lambda})} \qquad (10)$$

where,

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{n}{\alpha^2}$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n} \frac{(x_i/\lambda)^\beta \left(\log(x_i/\lambda)\right)^2}{\left\{1+(x_i/\lambda)^\beta\right\}^2}$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = \frac{n\alpha\beta}{\lambda^2} - \frac{(\alpha+1)\beta}{\lambda^2}\sum_{i=1}^{n} \frac{(x_i/\lambda)^\beta}{\left\{1+(x_i/\lambda)^\beta\right\}}$$
$$\qquad\qquad - \frac{(\alpha+1)\beta^2}{\lambda^2}\sum_{i=1}^{n} \frac{(x_i/\lambda)^\beta}{\left\{1+(x_i/\lambda)^\beta\right\}^2}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} \frac{\log(x_i/\lambda)}{\left\{1+(x_i/\lambda)^\beta\right\}}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = -\frac{n\beta}{\lambda} + \frac{\beta}{\lambda}\sum_{i=1}^{n} \frac{(x_i/\lambda)^\beta}{\left\{1+(x_i/\lambda)^\beta\right\}}$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \lambda} = -\frac{n\alpha}{\lambda} + \frac{(\alpha+1)}{\lambda}\sum_{i=1}^{n} \frac{(x_i/\lambda)^\beta}{\left\{1+(x_i/\lambda)^\beta\right\}}$$
$$\qquad\qquad + \frac{\beta(\alpha+1)}{\lambda}\sum_{i=1}^{n} \frac{(x_i/\lambda)^\beta \log(x_i/\lambda)}{\left\{1+(x_i/\lambda)^\beta\right\}^2}$$

The diagonal elements of $\left(I(\hat{\underline{\theta}})\right)^{-1}$ provide the asymptotic variances for the parameters α, β and λ, respectively. The $100(1-\gamma)\%$ confidence intervals for α, β and λ can be constructed as

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{var(\hat{\alpha})}\ ;$$
$$\hat{\beta} \pm z_{\gamma/2}\sqrt{var(\hat{\beta})}\ \text{ and} \qquad (11)$$
$$\hat{\lambda} \pm z_{\gamma/2}\sqrt{var(\hat{\lambda})}$$

where $z_{\gamma/2}$ is the upper percentile of standard normal variate.

3.2 Bootstrap confidence intervals

In this section we propose the confidence intervals based on the bootstrapping. Bootstrap methods are widely used to improve estimators or to build confidence intervals for the parameters. Usually, they provide estimators with smaller standard errors, and confidence intervals with a coverage level closer to the nominal level than confidence intervals obtained by applying asymptotic results. The percentile bootstrap (*Boot-p*) method, proposed by Efron and Tibshirani (1986), is widely used in practice. We have used the parametric bootstrap method to construct confidence intervals for the parameters as well as the reliability and hazard functions. To estimate the '*Boot-p*' confidence interval, we proceed as follows, Soliman et al.(2012):

Step 1. From the original data $\underline{x} = (x_1,...,x_n)$ compute the ML estimates of the parameters: α, β and λ by solving the nonlinear equations (9).

Step 2. Using the values of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ in (3) and (4) with some mission time t, we obtain the ML estimates $\hat{R}(t)$ and $\hat{h}(t)$ of the reliability and hazard functions.

Step 3. Use $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ to generate a bootstrap sample $\underline{x}^* = (x_1^*,...,x_n^*)$ of size n from (2) by using (6) .

Step 4. As in step 1, based on $\underline{x}^* = (x_1^*,...,x_n^*)$ compute the bootstrap sample estimates of α, β, λ, $R(t)$ and $h(t)$, say $\hat{\alpha}^*, \hat{\beta}^*, \hat{\lambda}^*, \hat{R}^*(t)$ and $\hat{h}^*(t)$.

Step 5. Repeat steps 3 and 4 N times representing N bootstrap MLE's of $(\alpha, \beta, \lambda, R(t), h(t))$ based on N bootstrap samples.

Step 6. Arrange all $\hat{\alpha}^*$'s, $\hat{\beta}^*$'s , $\hat{\lambda}^*$'s, $\hat{R}^*(t)$'s and $\hat{h}^*(t)$'s in an ascending order to obtain the bootstrap sample $\{\psi_k^1,...,\psi_k^N\}$, $k=1,2,3,4,5$, where $\psi_1 \equiv \hat{\alpha}^*$, $\psi_2 \equiv \hat{\beta}^*$, $\psi_3 \equiv \hat{\lambda}^*$, $\psi_4 \equiv \hat{R}(t)^*$ and $\psi_4 \equiv \hat{h}(t)^*$.

Let $G(z) = P(\psi_k \leq z)$ be the cumulative distribution function of ψ_k. Define $\psi_{kboot} = G^{-1}(z)$ for given z. The approximate bootstrap $100(1-\gamma)\%$ confidence interval of ψ_k is given by $\left(\psi_{kboot}(\gamma/2), \psi_{kboot}((1-\gamma)/2)\right)$.

4. Bayesian model formulation

Given a set of data $\underline{x} = (x_1, \ldots, x_n)$ from $ELL(\alpha, \beta, \lambda)$, the likelihood function is given by

$$L(\alpha, \beta, \lambda \mid \underline{x}) = \left(\frac{\alpha\,\beta}{\lambda}\right)^n \prod_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\alpha\beta-1} \prod_{i=1}^{n}\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{-(\alpha+1)}$$

The Bayesian model is constructed by specifying the prior distributions for the model parameters α, β and λ, and then multiplying with the likelihood function $L(\alpha, \beta, \lambda \mid \underline{x})$ for the given data $\underline{x} = (x_1, \ldots, x_n)$ to obtain the posterior distribution function using Bayes theorem. Denote the prior distribution of α, β and λ as $p(\alpha, \beta, \lambda)$. The joint posterior is

$$p(\alpha, \beta, \lambda \mid \underline{x}) \propto L(\alpha, \beta, \lambda \mid \underline{x})\, p(\alpha, \beta, \lambda)$$

Priors for the parameters

We assume the independent priors for the parameters. We have assumed informative priors for the shape parameters α and β whereas non-informative prior for the scale parameter λ. Non-informative prior refers to a prior for which we only know certain parameters' value ranges or their importance; for example, there may be a uniform distribution. Let us suppose the gamma priors for $\alpha \sim G(a_1, b_1)$, $\beta \sim G(a_2, b_2)$ and uniform prior for $\lambda \sim U(a_3, b_3)$ as

$$p(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)}\; \alpha^{a_1-1}\, \exp(-b_1\,\alpha)\; ; \alpha > 0,\; a_1 > 0,\; b_1 > 0$$

$$p(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)}\; \beta^{a_2-1}\, \exp(-b_2\,\beta)\; ; \beta > 0,\; a_2 > 0,\; b_2 > 0$$

and

$$p(\lambda) = \frac{1}{b_3 - a_3}\; ;\quad a_3 < \lambda < b_3.$$

Thus, we have

$$p(\alpha, \beta, \lambda) = p(\alpha)\, p(\beta)\, p(\lambda)$$

Posterior distribution

The expression for the posterior can be obtained, up to proportionality, by multiplying the likelihood with the prior and this can be written as

$$p(\alpha, \beta, \lambda \mid \underline{x}) \propto \alpha^{a_1+n-1}\, \beta^{a_2+n-1}\, \lambda^{-n}\, e^{-(b_1\alpha+b_2\beta)}\; T_1 \quad (12)$$

where

$$T_1 = \prod_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\alpha\beta-1} \prod_{i=1}^{n}\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{-(\alpha+1)} \quad (13)$$

The posterior is complicated and no close form inferences are possible. We, therefore, propose to consider MCMC methods to simulate samples from the posterior so that sample-based inferences can be easily drawn. To implement MCMC calculations, Markov chains require a stationary distribution. There are many ways to construct these chains. During the last decade, the following Monte Carlo (MC) based sampling methods for evaluating high dimensional posterior integrals have been developed: MC importance sampling, Metropolis-Hastings sampling, Gibbs sampling, and other hybrid algorithms. Gibbs sampling (Gelfand and Smith, 1990), the most popular MCMC sampling algorithm in the Bayesian computation literature, which is actually a special case of Metropolis-Hastings sampling, Metropolis et al. (1953) and Hastings (1970).

Gibbs Sampler : Algorithm

Gibbs sampling belongs to the Markov update mechanism and adopts the ideology of "divide and conquer." By using Gibbs sampling, we only need to know the full conditional distribution. Therefore, it is more advantageous in high-dimensional numerical computation. Gibbs sampling is essentially a special case of Metropolis-Hastings sampling, as the acceptance probability equals one. It is currently the most popular MCMC sampling algorithm in the Bayesian reliability inference literature. To carry out Gibbs sampling, the basic scheme is as follows:

Step1: Compute the posterior distribution, upto proportionality, and specify the full conditionals, using (12), of the model parameters α, β and λ as

- full conditional of α given β, λ and $\underline{x}$:

$$p(\alpha \mid \beta, \lambda, \underline{x}) \propto \alpha^{a_1+n-1}\; e^{-(b_1\alpha)}\; T_1$$

- full conditional of β given α, λ and $\underline{x}$

$$p(\beta \mid \alpha, \lambda, \underline{x}) \propto \beta^{a_2+n-1}\; e^{-(b_2\beta)}\; T_1$$

- full conditional of λ given α, β and $\underline{x}$:

$$p(\lambda \mid \alpha, \beta, \underline{x}) \propto \lambda^{-n}\; T_1$$

Step 2: Select an initial value $\underline{\theta}^{(0)} = \left(\alpha^{(0)}, \beta^{(0)}, \lambda^{(0)}\right)$ to start the chain.

Step 3: Suppose at the ith-step, $\underline{\theta} = (\alpha, \beta, \lambda)$ takes the value $\underline{\theta}^{(i)} = \left(\alpha^{(i)}, \beta^{(i)}, \lambda^{(i)}\right)$ then from full conditionals, generate

$$\alpha^{(i+1)} \text{ from } p\left(\alpha \mid \beta^{(i)}, \lambda^{(i)}, \underline{x}\right)$$

$$\beta^{(i+1)} \text{ from } p\left(\beta \mid \alpha^{(i+1)}, \lambda^{(i)}, \underline{x}\right) \text{ and}$$

$$\lambda^{(i+1)} \text{ from } p\left(\lambda \mid \alpha^{(i+1)}, \beta^{(i+1)}, \underline{x}\right).$$

Step 4: This completes a transition from $\underline{\theta}^{(i)}$ to $\underline{\theta}^{(i+1)}$

Step 5: Repeat Step 3, N times.

MCMC output : Posterior sample

Monitor the convergence using convergence diagnostics. Suppose that convergence have been reached after 'B' iterations (the burn-in period). Discard the observations $\left(\underline{\theta}^{(1)}, \underline{\theta}^{(2)}, \ldots, \underline{\theta}^{(B)}\right)$ and retain the observations

$$\left(\underline{\theta}^{(B+1+(j-1)l)}; B+1+(M-1)l \le N; j=1, 2, \ldots, M; l \ge 1\right)$$

which are viewed as being an independent sample from the stationary distribution of the Markov chain that is typically the posterior distribution, where l is the *lag* (or thin interval).

For the posterior analysis we have the MCMC output (posterior sample) $\left(\underline{\theta}^{(1)}, \ldots, \underline{\theta}^{(j)}, \ldots, \underline{\theta}^{(M)}\right)$, where

$$\underline{\theta}^{(j)} = \left(\alpha^{(j)}, \beta^{(j)}, \lambda^{(j)}\right); j=1,2,\ldots,M.$$

Thus MCMC output is referred as the sample after removing the initial iterations (produced during the burn-in period) and considering the appropriate lag.

The Bayes estimates of $\underline{\theta} = (\alpha, \beta, \lambda)$, under squared error loss function(SELF), are given by

$$\hat{\alpha} = \frac{1}{M}\sum_{j=1}^{M} \alpha^{(j)}; \ \hat{\beta} = \frac{1}{M}\sum_{j=1}^{M} \beta^{(j)}; \ \hat{\lambda} = \frac{1}{M}\sum_{j=1}^{M} \lambda^{(j)} \quad (14)$$

We shall use OpenBUGS software to obtain posterior samples. BUGS (Bayesian inference Using Gibbs Sampling) is a piece of computer software for the Bayesian analysis of complex statistical models using Markov chain Monte Carlo (MCMC) methods. OpenBUGS, with open source code, is a software package that implements MCMC algorithms and is able to analyse highly complex problems for the probability models available in OpenBUGS, Thomas et al.(2006). It is a powerful and flexible tool for Bayesian analysis. The modular framework of OpenBUGS provides an in depth and interactive analysis of the model with many built-in features and model extensions can easily be accommodated. Each new model (probability distribution) causes a new software system to be built. Model implementation is difficult for the probability distributions, which are not pre-defined in OpenBUGS. Several probability distributions useful in the field of reliability studies are incorporated into *OpenBUGS,* Kumar et al. (2010) and Lunn (2010).

As the three-parameter exponentiated log-logistic distribution is not available in OpenBUGS, it requires incorporation of a module in *ReliaBUGS,* Lunn et al.(2013), which is subsystem of *OpenBUGS*. A module *dexpo.loglogistic3_T(alpha, beta, lambda)* is written for three-parameter exponentiated log-logistic, the corresponding computer program can be obtained from authors, to perform full Bayesian analysis in OpenBUGS using the method described in Kumar et al. (2010), Kumar (2010) and Chaudhary and Kumar (2013b).

5. Data

The following real data set is considered for illustration of the proposed methodology. The data extracted from Ghitany et al. (2008), gives waiting times of 100 bank customers in (min)

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5

5.1. Computation of MLE and Model Validation

The maximum likelihood estimates (MLEs) are obtained by direct maximization of the log-likelihood function $\ell(\alpha, \beta, \lambda)$ given in (8).
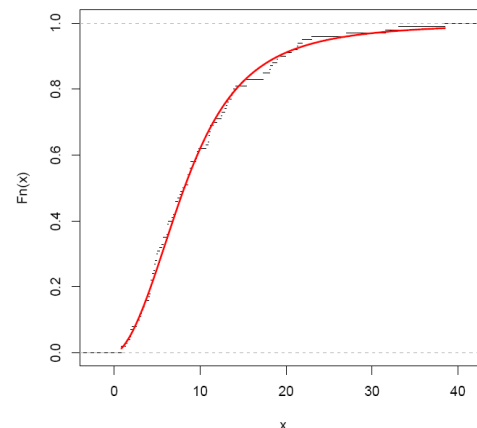

Figure 3: The graph of empirical and fitted distribution function.

The advantage of this procedure is that it runs immediately using existing statistical packages such as R software (R Development Core Team, 2013). We consider the Newton-Raphson algorithm in R (Rizzo, 2008) to compute the MLEs.

Table 1. MLE, standard error and 95% confidence interval (CI)

Parameter	MLE	Std. Error	95% CI
alpha	0.5787	0.1996	(0.1874, 0.9699)
beta	2.8731	0.5326	(1.8292, 3.9170)
lambda	10.9026	2.1506	(6.6874, 15.1179)

The Table 1 shows the ML estimates, standard error(SE) and 95 % Confidence Intervals for parameters α, β and λ. The value of loglikelihood is $\ell(\hat{\alpha}, \hat{\beta}, \hat{\lambda}) = -318.384$. The Akaike information criterion (AIC) and Bayesian information criterion(BIC) can be used to determine which model is most appropriate for the given data. For the given data set AIC = 642.769 and BIC = 650.584.
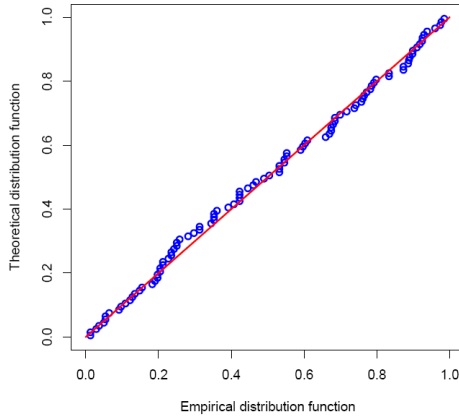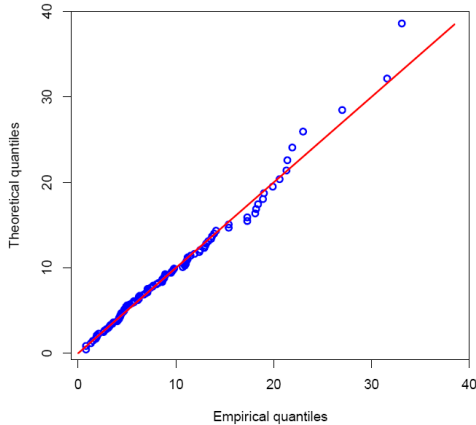

Figure 4: Probability-Probability(P-P) plot using MLEs as estimate

Figure 5: Quantile-Quantile(Q-Q) plot using MLEs as estimate.

We compute the Kolmogorov-Smirnov (KS) distance between the empirical distribution function and the fitted distribution function when the parameters are obtained by method of maximum likelihood to check the validity of the model. The value of KS statistic is 0.052 and the corresponding *p*-value is 0.95.

The high *p*-value suggests that fit is satisfactory. We have plotted the empirical distribution function and the fitted distribution function in Figure 3. The P–P and Q–Q plots, Kumar and Ligges(2011), for the fitted model are shown in Figure 4 and Figure 5. It can be seen that the fitted exponentiated log-logistic distribution provides good fit to the given data.

6. Bayesian Analysis

We assume the independent gamma priors for $\alpha \sim G(a_1, b_1)$, $\beta \sim G(a_2, b_2)$ and uniform prior for $\lambda \sim U(a_3, b_3)$ with hyper-parameter values $(a_1 = b_1 = a_2 = b_2 = 0.001)$ and $(a_3 = 0.0, b_3 = 50.0)$. We run the program code given as in Script 1 into OpenBUGS to generate two Markov chains at the length of 40,000 with different starting points of the parameters. We have chosen initial values for the parameters, wide spread over the parameter space, $(\alpha = 0.5, \beta = 0.5, \lambda = 5.0)$ for the first chain and $(\alpha = 3.0, \beta = 2.5, \lambda = 25.0)$ for the second chain. The convergence is monitored using trace BGR plots. It can be observed that the Markov chains reached to the stationary condition very quickly, approximately 2000 iterations. Therefore, burn-in of 5000 samples is more than enough to erase the effect of starting point(initial values). Finally, samples of size 7000 are formed from the posterior by picking up equally spaced every fifth outcome (to minimize the auto correlation among the generated deviates.), i.e. thin=5, starting from 5001.

Script 1: OpenBUGS *script for the Bayesian analysis*

```
model
{
    for( i in 1 : N )
    {
    x[i] ~ dexpo.loglogistic3_T(alpha, beta, lambda)
    f[i] <- density(x[i], x[i])
    reliability[i] <- R(x[i], x[i])
    }
# Prior distributions of the model parameters
    alpha ~ dgamma(0.001, 0.001)
    beta ~ dgamma(0.001, 0.001)
    lambda ~ dunif(0.0, 50.0)
}
Data
    list(N=100, c(0.80,...,38.5))
Initial values
    list(alpha= 0.5, beta=0.5, lambda= 5.0)   # Chain 1
    list(alpha=3.0, beta=2.5, lambda= 25.0)   # Chain 2
```

Therefore, we have the posterior sample from chain 1 and chain 2 as $\left(\alpha_i^{(j)}, \beta_i^{(j)}, \lambda_i^{(j)}\right)$; $j = 1, \ldots, 7000; i = 1, 2$. We have considered the chain 1 for analysis as well as for convergence diagnostic plots, the chain 2 produces the similar result.

6.1 Convergence diagnostics

Because of the Markov chain's ergodic property, all inferences are implemented under the assumption that the Markov chain converges. Therefore, the Markov chain

Monte Carlo convergence diagnostic is very important. The inferences are valid, if the simulated sample provides a reasonable approximation for the posterior density. Fundamental graphical diagnostics are the *trace* and the Brooks-Gelman-Rubin(BGR) plots. Trace plots, separately for each model parameter, are one of the oldest methods of qualitatively assessing MCMC sampler performance. The MCMC iteration number is on the x-axis and the value of the parameter drawn at each iteration is on the y-axis successive values are joined by a line. When more than one chain has been run, the lines from all chains are plotted in different colors in the same panel.

Figure 6(left panel) shows the sequential realizations of the parameters of the model for chain 1. They look like white noise-just random swinging around in that range, without any consistent pattern. These are hallmarks of rapid MCMC convergence. Thus Markov chain seems to be mixing well enough and is likely to be sampling from the stationary distribution.
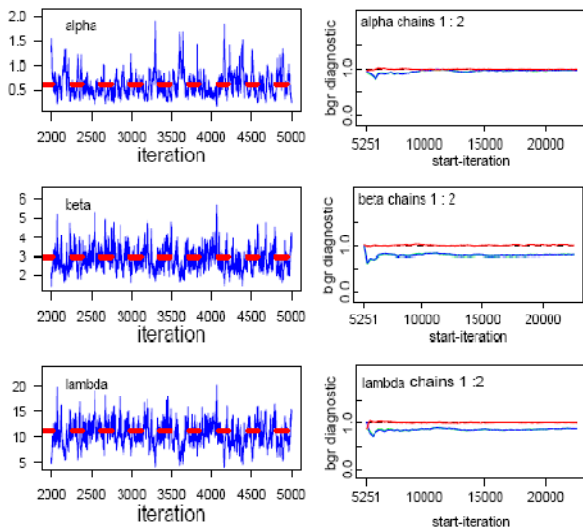
Figure 6: The trace plot (left panel) and the Brooks-Gelman-Rubin(BGR) plot(right panel) for alpha, beta and lambda.

The BGR convergence diagnostics is based on running multiple parallel chains from over-dispersed starting values. The convergence is assessed through an ANOVA-type diagnostic test by calculating and comparing within- and between-chain variability. General idea is to test if variability within the chain is higher than between chains.

OpenBUGS includes the Brooks-Gelman-Rubin (BGR) convergence statistic. The chains for a model parameter are judged to have converged if the ratio of between to within variability is close to one. The green line represents the between variability, the blue line represents the within variability, and the red line represents the ratio. Evidence for convergence comes from the red line being

close to 1 on the y-axis and from the blue and green lines being stable (horizontal) across the width of the plot.

Figure 6(right panel) shows the BGR plots for the model parameters. There is no evidence that our posterior samples produced by OpenBUGS chains failed to converge, so we can proceed to use posterior samples for Bayesian inference.

6.2 Posterior analysis
6.2.1 Numerical summary

The numerical summary is presented for $\left(\alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}\right)$; $j = 1,\dots,7000$ from chain 1.

We have considered various quantities of interest and their numerical values based on MCMC sample of posterior characteristics for exponentiated log-logistic distribution. The MCMC results of the posterior mean, mode, standard deviation(SD), five point summary statistics (minimum, first quartile, median, third quartile and maximum), 2.5th percentile, 97.5th percentile, and skewness of the parameters α, β and λ are displayed in Table 2.

Table 2. Numerical summaries based on MCMC sample of posterior characteristics

Characteristics	alpha	beta	lambda
Mean	0.5795	2.9912	11.3276
Standard Deviation	0.2010	0.6090	2.3417
Minimum	0.1462	1.4520	4.1020
2.5th Percentile($P_{2.5}$)	0.2664	2.0410	6.9808
First Quartile (Q_1)	0.4316	2.5500	9.7085
Median	0.5506	2.9075	11.2000
Third Quartile (Q_3)	0.7005	3.3350	12.8700
97.5th Percentile($P_{97.5}$)	1.0495	4.4551	16.2700
Maximum	1.2510	5.6440	20.8800
Mode	0.4828	2.7750	10.9166
Skewness	0.6746	0.7947	0.2417

Highest probability density (HPD) : The HPD credible intervals for α, β and λ can be constructed by using algorithm given in Chen and Shao (1999).

Let $\left\{\alpha_{(j)}; j=1,2,\dots,M\right\}$ be the corresponding ordered MCMC sample of $\left\{\alpha^{(j)}; j=1,2,\dots,M\right\}$. Then, the $100(1-\gamma)\%$ HPD intervals for α is

$$\left(\alpha_{(k^*)}, \alpha_{\left(k^*+\left[(1-\gamma)M\right]\right)}\right),$$

where k^* is chosen so that

$$\alpha_{\left(k^{*}+\left[(1-\gamma)M\right]\right)} - \alpha_{(k^{*})} = \min_{1 \le k \le M - \left[(1-\gamma)M\right]} \left(\alpha_{\left(k+\left[(1-\gamma)M\right]\right)} - \alpha_{(k)} \right) .$$

Here $[\chi]$ denotes the largest integer less than or equal to χ. In the same fashion, one can also obtain the Bayes HPD credible intervals for the other parameters. Table 3 shows the symmetric credible intervals(SCI) and HPD credible intervals for parameters alpha, beta and lambda. We have also computed the 95% bootstrap confidence interval (BCIs), using the algorithm of the percentile bootstrap method, described in section 3.2, we present the mean of 1000 bootstrap samples of the parameters.

Table 3. Two-sided 95% intervals

Parameter	SCI	HPD	BCI
alpha	(0.266, 1.050)	(0.231, 0.992)	(0.310, 1.150)
beta	(2.041, 4.455)	(1.901, 4.212)	(2.118, 4.229)
lambda	(6.98, 16.27)	(6.743, 15.950)	(6.865, 14.699)

6.2.2 Visual summary

The visual graphs include the boxplot, density strip plot, histogram, marginal posterior density estimate and rug plots for the parameters. We have also superimposed the 95% HPD intervals.

These graphs provide almost complete picture of the posterior uncertainty about the parameters. We have used the posterior sample $\left(\alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)} \right)$; $j = 1, \dots, 7000$ to draw these graphs.
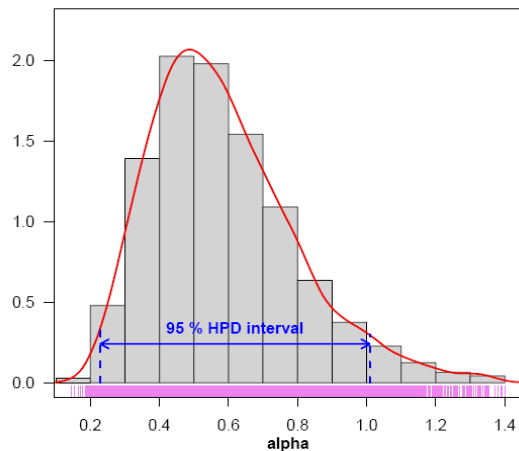
Figure 7(a): Histogram, marginal posterior density and 95% HPD interval based on posterior sample.

The density strip plot introduced by Jackson (2008) for a univariate distribution as a shaded rectangular strip, whose darkness at a point is proportional to the probability density. It may be noted that density strip plot is more informative as compared to boxplot.
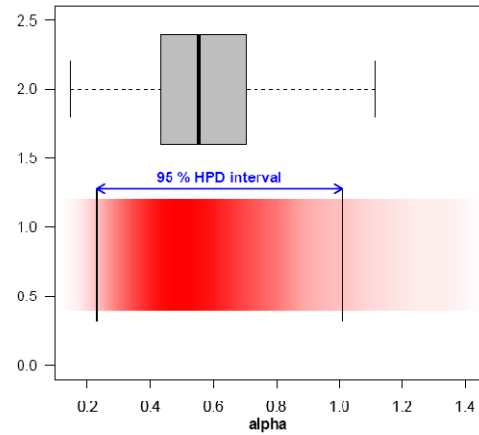

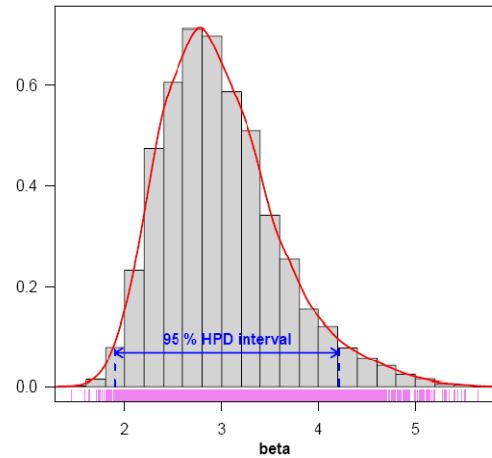*Figure 7(b):* Boxplot and density strip plot of α, based on posterior sample.

Figure 8(a): Histogram, marginal posterior density and 95% HPD based on posterior sample.
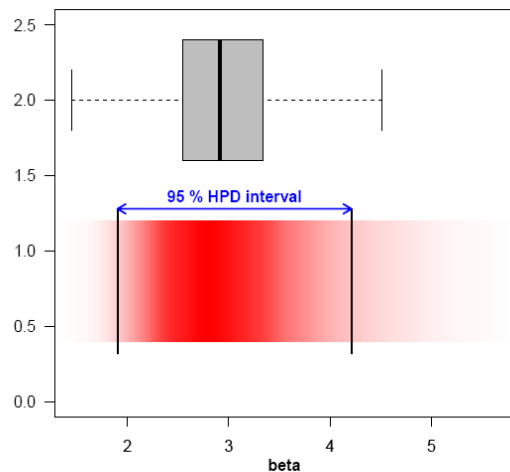

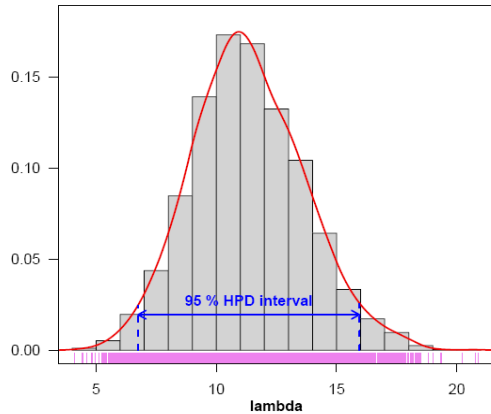*Figure 8(b):* Boxplot and density strip plot of beta

Figure 9(a): Histogram, marginal posterior density and 95% HPD interval based on posterior sample.

Probability histograms are the most popular non-parametric method to estimate the density function. It gives an idea about skewness, behaviour in the tails, presence of multi-modal behaviour, and data outliers. It may be useful to compare the fundamental shapes associated with standard analytic distributions.

The kernel density estimates have been drawn using R software with the assumption of Gaussian kernel and properly chosen values of the bandwidths. It can be seen that λ is symmetric whereas α and β show positive skewness.

Figure 7(a) represents the histogram, marginal posterior density, rug plot and 95% HPD interval for α. The boxplot and the density strip plot are displayed in Figure 7(b).
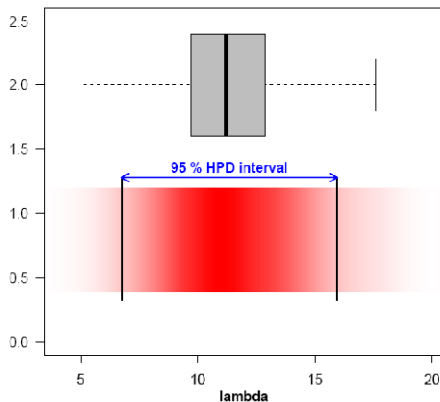

Figure 9(b): Boxplot and density strip plot of λ

We have plotted the similar graphs in Figure 8(a) and (b) for β. The plots for λ are shown in Figure 9(a) and Figure 9(b).

6.2.3 Comparison with MLE

For the comparison with MLE we have plotted three graphs. In Figure 10 the density functions $f(x; \hat{\alpha}, \hat{\beta}, \hat{\lambda})$ using MLEs and Bayesian estimates, computed via MCMC samples under gamma priors for α and β and uniform prior for λ, are plotted.
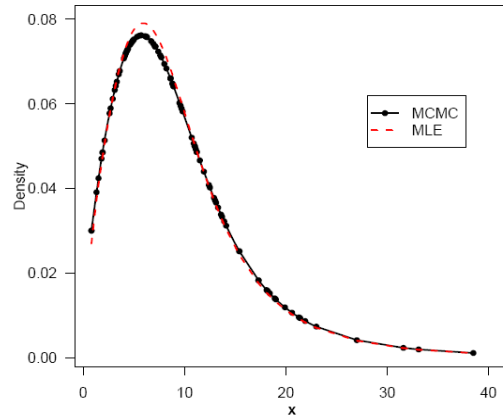

Figure 10: The density functions using MLEs and Bayesian estimates, computed via MCMC.

The density corresponding to MLE has been plotted using the "plug-in" estimates of the parameters. We observe in the Figure 10, the MLEs and the Bayes estimates are quite close and fit the data very well.

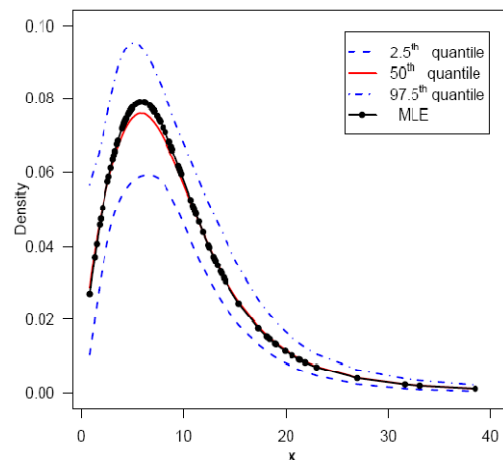A further support for this finding can be obtained by inspecting the Figure 11.

Figure 11: Density estimates

In Figure 11 we have plotted $2.5^{\text{th}}, 50^{\text{th}}$ and 97.5^{th} quantiles of the estimated density, it can be considered as evaluation of model fit, based on posterior sample,

$$\left(\alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}\right); j = 1, \ldots, 7000 .$$

We have computed the density function at each observed data point for 7000 posterior samples,

$$f\left(x_i; \alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}\right); i = 1, \ldots, 100 .$$

6.2.4 Estimation of hazard and reliability functions

The posterior samples may be used to completely summarize the posterior uncertainty about the parameters α, β and λ. This is also true of any function of the parameters e.g. reliability and hazard functions. Suppose we wish to give point and interval estimates for reliability and hazard functions at the mission time $t=4.1$ (at the 19th observed data point). We have computed the hazard and reliability functions at mission time $t=4.1$ (at the 19th observed data point) for 7000 posterior samples, using logical function $hrf()$ and $reliability(\ )$, (Kumar et al. 2010) in OpenBUGS.
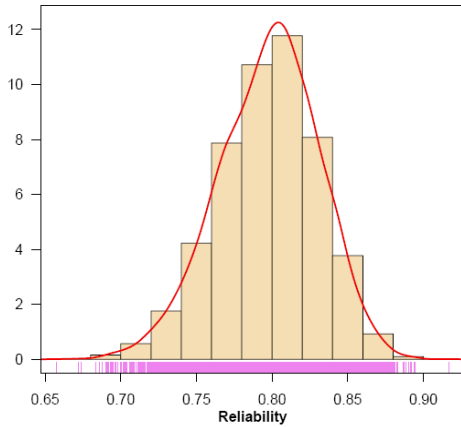

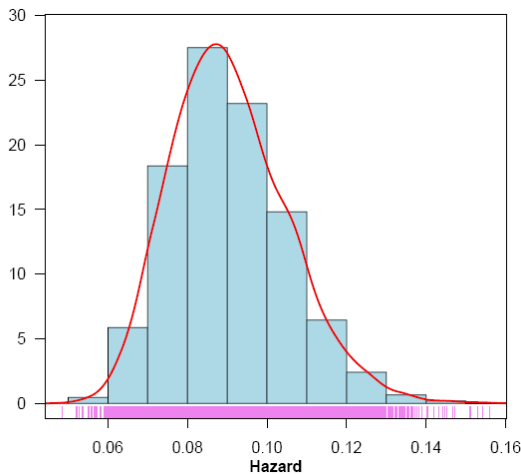***Figure 12(a):*** Visual summary of reliability function at $t=4.1$

Figure 12(b): Visual summary of hazard function at $t=4.1$

It can be computed directly using hazard and reliability functions given in (4) and (3), respectively.

$$h\left(x = 4.1; \alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}\right); j = 1, \ldots, 7000 \quad \text{and}$$

$$R\left(x = 4.1; \alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}\right); j = 1, \ldots, 7000 .$$

The histogram and marginal posterior density of the reliability and hazard functions are shown in Figure 12(a) and Figure 12(b) respectively based on samples of size 7000. The Gaussian kernel has been used for kernel density estimates.

It is evident from Figure 12 that the marginal distribution of reliability is negatively skewed whereas hazard is positively skewed.
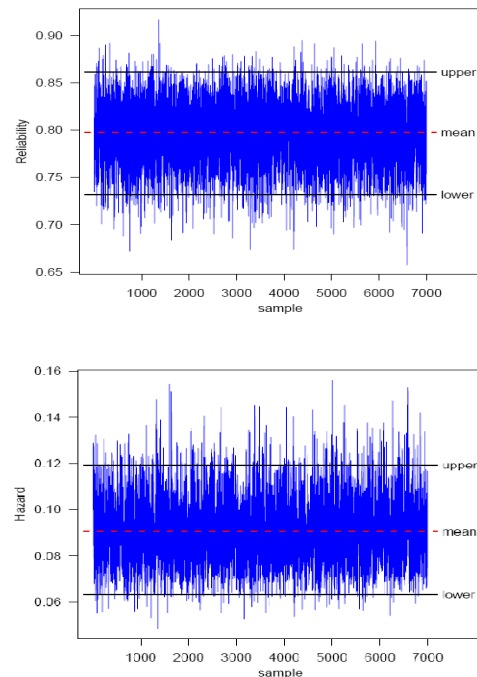

Figure 13: MCMC output of $R(t = 4.1)$ and $h(t = 4.1)$. Dashed line(...) represents the posterior median and solid lines(-) represent lower and upper bounds of 90% probability intervals (HPD)

A trace plot is a plot of the iteration number against the value of the draw of the parameter at each iteration. Figure 13 display 7000 chain values for the hazard $h(t = 4.1)$ and reliability $R(t = 4.1)$ functions, with their sample median and 90% credible intervals.

The MCMC results of the posterior mean, mode, standard deviation(SD), five point summary statistics (minimum, first quartile, median, third quartile and maximum), 2.5^{th} percentile, 97.5^{th} percentile, skewness, 95% symmetric credible intervals(SCI) and HPD credible intervals of reliability and hazard functions are displayed in Table 4.

Table 4. Posterior summary for Reliability and Hazard functions at $t=4.1$

Characteristics	Reliability	Hazard
Mean	0.7975	0.0906
Standard Deviation	0.0334	0.0146
Minimum	0.6575	0.0485
2.5th Percentile($P_{2.5}$)	0.7279	0.0656
First Quartile (Q_1)	0.7752	0.0801
Median	0.7995	0.0892
Third Quartile (Q_3)	0.8208	0.0998
97.5th Percentile($P_{97.5}$)	0.8583	0.1226
Maximum	0.9167	0.1559
Mode	0.8044	0.0874
Skewness	-0.2720	0.4930
95% SCI	(0.7279, 0.8583)	(0.0656, 0.1226)
95% HPD	(0.7316, 0.8612)	(0.0633, 0.1192)

The ML estimates of reliability and hazard function at $t=4.1$ are computed using invariance property of the MLE. We have $\hat{h}(t=4.1)=0.0915$ and $\hat{R}(t=4.1)=0.8098$.

The 95% percentile bootstrap confidence interval (BCIs) for reliability and hazard function at $t=4.1$, using the algorithm described in section 3.2, based on 1000 bootstrap samples are $(0.6563, 0.9982)$ and $(0.0520, 0.2240)$, respectively.

Now we shall demonstrate the effectiveness of proposed methodology for the entire data set. For this, we have estimated the reliability function using posterior samples. Since we have an effective MCMC technique, we can estimate any function of the parameters. We have used the Kaplan-Meier estimate of the reliability function to make the comparison more meaningful.

$$\hat{R}(x_i) = \frac{1}{M} \sum_{j=1}^{M} R\left(x_i \mid \alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}\right); \; i=1,\ldots,n$$

$$\hat{h}(x_i) = \frac{1}{M} \sum_{j=1}^{M} h\left(x_i \mid \alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}\right)$$

Here we have $i=100$ and $M=7000$.

The Figure 14, exhibits the estimated reliability function (dashed line: 2.5^{th} and 97.5^{th} quantiles; solid line: 50th quantile) using Bayes estimate based on MCMC output and the empirical reliability function (solid step-line). The Figure 14 shows that reliability estimate based on MCMC is very close to the empirical reliability estimates.

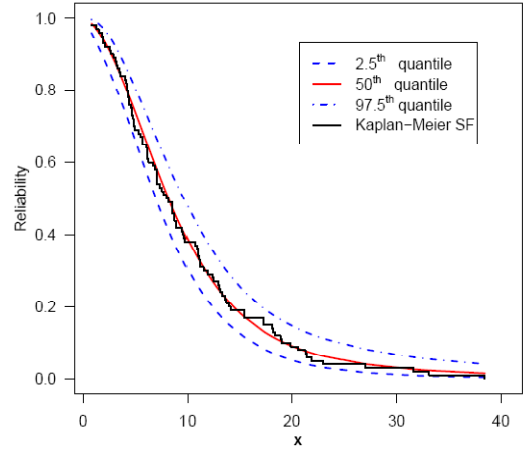We have plotted the estimated hazard function using Bayes estimate based on MCMC in Figure 15.

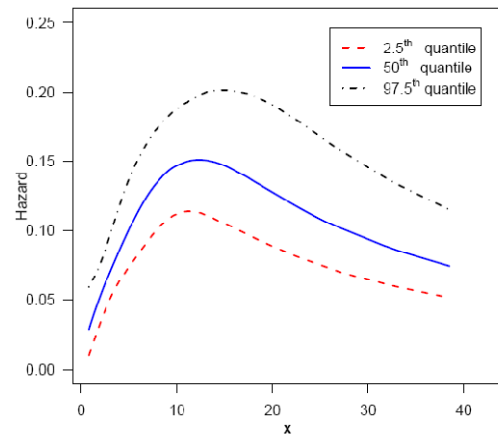**Figure 14:** Reliability function estimate using MCMC and Kaplan-Meier estimate

Figure 15: Hazard function estimate using MCMC

7. Posterior predictive analysis

One of the best and most flexible approaches to examining model fit is the use of posterior predictive distributions, (Gelman 2003) and (Gelman et al. 2004).

To evaluate the fit of the posterior distribution of a Bayesian model, we can compare the observed data to the posterior predictive distribution. Gelman et al. (1996) propose a diagnostic procedure known as a posterior predictive checking using predictive replicates. Various forms of checking function may be calculated for both new data and actual observations to assess whether the model satisfactorily reproduces certain important aspects of the actual data. Thus such checks go beyond bias and precision.

The posterior predictive distribution is either the replication of observation x given the model (usually represented as x^{rep}), or the prediction of a new and unobserved x (usually represented as x^{new}), given the

model. This is the likelihood of the replicated or predicted data, averaged over the posterior distribution

$$p\left(x^{rep}|\underline{x}\right) = \int p\left(x^{rep}|\underline{x}\right) p\left(\underline{\theta}\,|\,\underline{x}\right) d\underline{\theta}$$

Modern Bayesian computational tools however provide straightforward solutions as one can easily simulate predictive samples if MCMC outputs are available from the posterior corresponding to the assumed model.

Most of the standard numerical and graphical methods based on predictive distribution can then be easily implemented to study the compatibility of the model. Comparing the predictive distribution x^{rep} to the observed data x is generally termed a "posterior predictive check".

The graphical method is one of the best way to assess model adequacy based on posterior predictive distributions.

Implementation of posterior predictive simulation is relatively simple, given an MCMC-generated sample of size 2000 from the posterior distribution for the parameters in a model $\underline{\theta} = (\alpha, \beta, \lambda)$, and can often be incorporated as part of the MCMC algorithm itself. For each value of $\underline{\theta}$ simulated from the posterior, we generate a new observation from the sampling distribution for the data, using that parameter value, for every original observation in the sample. Thus, we have

$$x_i^{rep,(j)} \;; i = 1,\ldots,100$$

for each $\left(\alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}\right) \;\; j = 1,\ldots, 2000$.

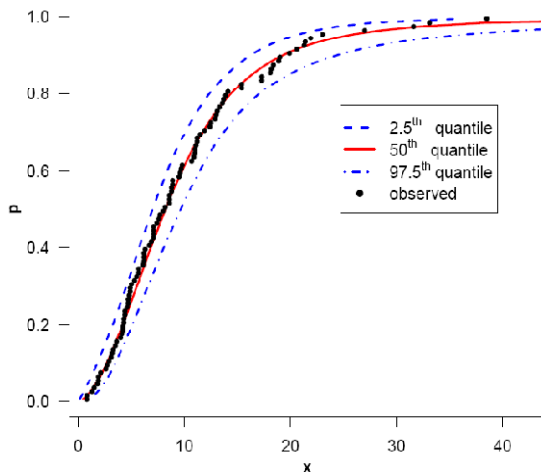In fact, we have predicated the entire data set and we have 2000 replications of each x_i ; $i = 1,\ldots,100$.

We view the model-checking as a comparison of the data with the replicated data given by the model, which includes exploratory graphics, Shrestha and Kumar(2013). The estimate of CDF based on replicated data given by the model is displayed in Figure 16. In fact statistical graphics provides implicit or explicit model checks. Figure 17 represents the Q-Q plot of predicted quantiles *vs.* observed quantiles. We, therefore, conclude that the exponentiated log-logistic distribution is compatible with the given data set.
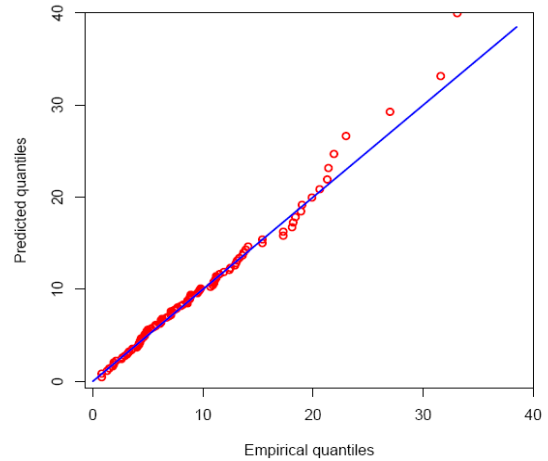

Figure 17: Q-Q plot of predictive quantiles versus empirical quantiles

Figure 18 exhibits graphical posterior predictive check of the model adequacy, solid line( ) represents the posterior median and dashed lines(...) represent lower and upper bounds of 95% probability intervals, observed data is superimposed.
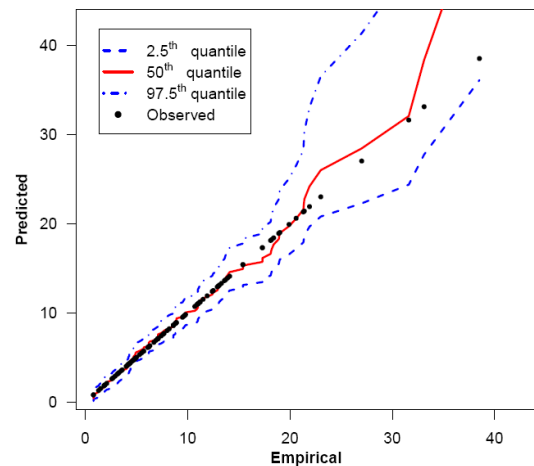

Figure 16: Estimate of CDF based on predicted values

Figure 18: Graph of model fit

To obtain further clarity on our conclusion for the study of model compatibility, we have considered

plotting of density estimates of second smallest, second largest and 19$^{\text{th}}$ smallest i.e. $\left(X_{(2)}, X_{(99)} \text{ and } X_{(19)} \right)$ replicated future observations from the model with superimposed corresponding observed data. For this purpose, 2000 samples have been drawn from the posterior using MCMC procedure and then obtained predictive samples from the model under consideration using each simulated posterior sample.
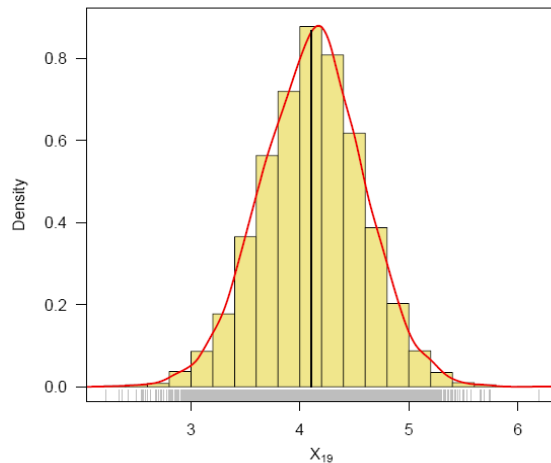

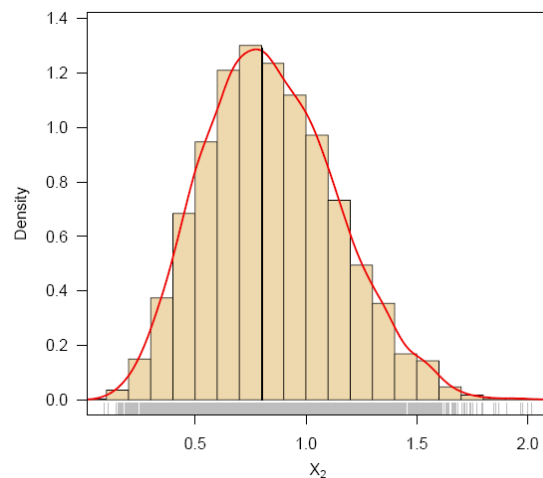***Figure 19:*** Posterior predictive density of $X_{(19)}$, vertical line represents corresponding observed value

Figure 20: Posterior predictive distribution of the second smallest $\left(x_{(2)} \right)$, vertical lines represent corresponding observed values

The predictive data reflect the expected observations after replicating the experiment in future, having already observed $\underline{x}$ and assuming that the adopted model is true. Overall, the results of the posterior predictive simulation indicate that model fits these data particularly well. Model fit assessments based on posterior predictive checks should not be used for model selection, Ntzoufras (2009).

The Table 5 shows the MCMC results of the posterior mean, median, mode and 95% HPD credible intervals for $\left(X_{(2)}, X_{(3)}, X_{(19)}, X_{(80)}, X_{(98)}, X_{(99)} \text{ and } X_{(100)} \right)$.

Table 5. Posterior characteristics

	Observed	Mean	Median	Mode	95% HPD
$X_{(2)}$	0.80	0.84	1.04	0.78	(0.28, 1.41)
$X_{(3)}$	1.30	1.14	1.38	1.10	(0.49, 1.80)
$X_{(19)}$	4.10	4.12	4.43	4.17	(3.14, 4.97)
$X_{(80)}$	13.90	14.31	15.07	13.89	(12.07, 16.63)
$X_{(98)}$	31.60	33.10	36.26	29.90	(23.03, 45.46)
$X_{(99)}$	33.10	39.91	44.35	35.19	(25.12, 56.91)
$X_{(100)}$	38.50	59.57	68.02	49.90	(32.38, 95.44)

The posterior predictive density estimates based on replicated future data sets are shown in Figure 19 and Figure 20 and Figure 21. Figure 20 represents the estimates corresponding to second smallest and Figure 21 second largest predictive observations, whereas the same for 19$^{\text{th}}$ smallest observation is shown in Figure 19. The corresponding observed values are also shown by means of vertical lines.
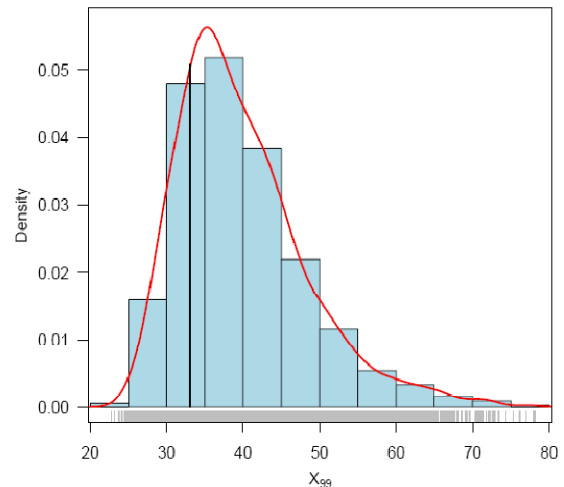

Figure 21: Posterior predictive distribution of the $\left(x_{(99)} \right)$, vertical lines represent corresponding observed value.

As the Figures 19, 20 and 21 show, the posterior predictive distributions are centered over the observed values, which indicate good fit. In general, the distribution of replicated data appears to match that of the observed data fairly well.

8. Conclusion

The methods described to build a full framework to accommodate academic research and engineering applications seeking to implement modern computational-based classical as well as Bayesian approaches related to exponentiated log-logistic distribution, especially in the area of reliability. We have proposed an integrated procedure for Bayesian inference using Markov chain Monte Carlo methods. For the sake of comparison, we have discussed the maximum likelihood estimation. The bootstrap percentile method has been used for the computation of confidence intervals of parameters and their functions. We have used exploratory data analysis techniques for the posterior analysis. We have shown that it is true for any function of the parameters such as hazard function, reliability etc. We have obtained the probability intervals for parameters, hazard and reliability functions. We have presented the model compatibility analysis via the posterior predictive check method. We have applied the developed techniques on a real data set.

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