International Journal of Statistika and Mathematika, ISSN: 2277- 2790 E-ISSN: 2249-8605, Volume 9, Issue 3, 2014 pp 82-84

Bayesian Analysis of Continuous Fertility Model

Himanshu Pandey*, Arun Kumar Rao, Ashivani Kumar Yadav#

Department of Mathematics and Statistics, D.D.U. Gorakhpur University, Gorakhpur, Uttar Pradesh, INDIA.

Corresponding Addresses:

*himanshu_pandey62@yahoo.com, #akyadavballia@gmail.com

Research Article

Abstract: The present paper aims at exploring a probability model of continuous fertility and also studies its Bayesian analysis under the precautionary loss function.

Keyword: Probability Model, Waiting Time, Conception, precautionary Loss Function.

Introduction

The study of women fecundability has controversially been adopted by different workers. The utility of the study depends upon the proper adoption of the fecundability of women. In most of the available literature, it is found that the fecaundability is assumed to be constant for all women [Singh (1964a), Pathak (1978)]. But in real life there are ample evidences that women vary in their fecundability. So, the fecundability may be thought of as a random variable {Henry (1995), Singh (1964)}. The present work deals with the same concept. Let us suppose that fecundability, say θ, follows distribution with p.d.f. g (θ). If T is waiting time for first conception can be treated as random variable which follows the distribution with p.d.f. f(x/θ) is regarded as a conditional p.d.f. of X for given θ where marginal probability density function of θ is given by g(θ) the study can be continued.

The Continuous Fertility Model

The geometric distribution is being considered as a discrete model for the waiting time first conception as developed by Gini (1924) the continuous model for the analysis of waiting time of first conception. The intuitive properties of exponential distribution also helped in such considerations. For such analysis Geometric distribution was replaced by the exponential distribution. Thus if x denotes the time of first conception, then its probability density function, say f(x;θ) is given by

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta};\ x > 0,\ \theta > 0 \qquad (2.1.1)$$

Where θ is instantaneous fecundability.

The survival function, say S (x) is given by

$$S(x) = P[X > x]$$

$$= \int_x^{\infty} \frac{1}{\theta} e^{-x/\theta} dx$$

$$= \frac{1}{\theta}\left[\frac{e^{-x/\theta}}{-\frac{1}{\theta}}\right]_x^{\infty}$$

$$\text{Or } S(x) = e^{-x/\theta} \qquad (2.1.2)$$

And the conception rate, say w(x) will be

$$W(x) = \frac{f(x)}{S(x)}$$

$$= \frac{\frac{1}{\theta} e^{-x/\theta}}{e^{-x/\theta}}$$

$$= \frac{1}{\theta} \qquad (2.1.3)$$

Maximum Likelihood Estimator

$$F(\underline{x}/\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$

$$= \left(\frac{1}{\theta}\right)^n e^{-z/\theta} \qquad (2.1.4)$$

Where $z = \sum_{i=1}^{n} x_i$

Bayesian Analysis of the Model

The first conception of the family is also a part of the past family back ground; therefore Bayesian analysis of conception seems realistic on the basis of some history. In the some coming section the Bayesian analysis has been done for a continuous time model i.e. exponential distribution.

We have

$$f(x/\theta) = \theta\, e^{-\theta x};\ x > 0,\ \theta > 0$$

Where θ is the instantaneous fecundibility.

The fundamental problems in Bayesian Analysis are that of the choice of prior distribution g(θ) and a loss function $L(\hat{\theta}, \theta)$. Let us consider three prior distribution of θ to obtain the Bayes estimators which are as follows:

(i) Quasi-Prior

For the situation where the experimenter has no prior information about the parameter θ, one may use the quasi density ass given by

$$g_1(\theta) = \frac{1}{\theta^d};\ \theta > 0,\ d > 0 \qquad (2.1.5)$$

Here d = 0 leads to a diffuse prior and d = 1, a non informative prior.

(ii) Natural Conjugate Prior of θ

The most widely used prior distribution of θ is the inverted gamma distribution with parameters α and β (>0) with p.d.f. given by

$$g_2(\theta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta};\ \theta > 0\ (\alpha,\beta) > 0 \\ 0;\ \text{otherwise} \end{cases} \qquad (2.1.6)$$

The main reason for general acceptability is the mathematical tractability resulting from the fact that inverted gamma distribution is conjugate prior for θ.

(iii) Uniform Prior

It Frequently happens that the life tester knows in advance that the probable values of θ lies over a finite range [α, β] but he does not have any strong opinion about any subset of values over this range. In such a case uniform distribution over [α, β] may be a good approximation.

$$g_3(\theta) = \begin{cases} \frac{1}{\beta-\alpha}; \ 0 < \alpha < \theta \leq \beta \\ 0 \ ; \ \text{otherwise} \end{cases} \quad (2.1.7)$$

Loss Function

The Bayes estimator $\hat{\theta}$ of θ is of course, optimal relative to the loss function chosen. A commonly used loss function is the squared error loss function (SELF)

$$L(\hat{\theta},\theta) = (\hat{\theta} - \theta)^2, \quad (2.1.8)$$

which is a symmetrical loss function and assigns equal losses to over estimation and underestimation. Canfield (1970) points out that the use of symmetric loss function may be inappropriate in the estimation of reliability function. Over estimation of reliability function or average lifetime is usually much more serious than under estimation of reliability function or mean failure time. Also, an under estimate of the failure rate results in more serious consequence than an overestimation of the failure rate. This leads to statistician to think about asymmetrical loss functions which have been proposed in statistical literature. It is well known that the Bayes estimator under the above loss function, say $\hat{\theta}$s, is the posterior mean. The squared error loss function (SELF) is often used also because it does not lead to extensive numerical computation but several authors {Ferguson (1967), Varian (1975), Berger (1980), Zellner (1986) and Basu and Ebrahimi (1991)} have recognized the inappropriateness of using symmetric loss function in several estimation problems. These have proposed different asymmetric loss function.

Precautionary Loss Function

Norstrom (1996) introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss function with quadratic loss function as a special case. These loss function approach infinitely near the origin to prevent underestimation and thus giving a conservative estimators, especially when low failure rates are being estimated. These estimators are very useful when under estimation may lead to serious consequences. A very useful and simple asymmetric precautionary loss function is

$$L(\hat{\theta},\theta) = \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}} \quad (2.1.9)$$

The posterior expectation of loss function in (2.1.18) is

$$E_\pi\left[L(\hat{\theta},\theta)\right] = E_\pi\left(\frac{\theta^2}{\hat{\theta}}\right) + E_\pi(\hat{\theta}) - 2E_\pi(\theta) \quad (2.1.10)$$

The value of $\hat{\theta}$ that minimises (2.1.10), denoted by $\hat{\theta}_P$, Bayes estimater of θ under Precautionary loss function is obtained by solving the following equation

$$\frac{d}{d\theta} E_\pi\left[L(\hat{\theta},\theta)\right] = 0$$

$$\Rightarrow \left[E_\pi\left\{\theta^2\left(-\frac{1}{\hat{\theta}_P{}^2}\right)\right\} + E_\pi(1)\right] = 0$$

$$\Rightarrow \left(-\frac{1}{\hat{\theta}_P{}^2}\right) E_\pi(\theta^2) = -1$$

$$\Rightarrow \hat{\theta}_P = \left[E_\pi(\theta^2)\right]^{\frac{1}{2}} \quad (2.1.11)$$

2. Bayes Estimator under $g_1(\theta)$

Under $g_1(\theta)$, the posterior distribution is defined by

$$f(\theta|\underline{x}) = \frac{f((\underline{x}|\theta))g_1(\theta)}{\int_0^\infty f((\underline{x}|\theta))g_1(\theta)d\theta} \quad (2.2.1)$$

Substituting the values of $g_1(\theta)$ and $f(\underline{x}|\theta)$ from equations (2.1.9) and (2.1.7) in (2.2.1) we get, after simplification, as

$$f(\theta|\underline{x}) = \frac{\left(\frac{1}{\theta}\right)^n\left(\frac{1}{\prod_{i=1}^n x_i}\right)e^{-\left(\frac{z}{\theta}\right)}\frac{1}{\theta^d}}{\int_0^\infty\left(\frac{1}{\theta}\right)^n\left(\frac{1}{\prod_{i=1}^n x_i}\right)e^{-\left(\frac{z}{\theta}\right)}\frac{1}{\theta^d}d\theta} \quad (2.2.2)$$

$$= \frac{z^{n+d-1}}{\Gamma(n+d-1)}\theta^{-(n+d)}e^{-z/\theta}\ ; \theta>0, n+d>1.$$

The Bayes estimator under squared error loss function is the posterior mean given by $\hat{\theta}_S = \int_0^\infty \theta \ f(\theta|\underline{x})d\theta$. (2.2.3) Substituting the values of $f(\theta|\underline{x})$ from equation (2.2.2) in equation (2.2.3) and on solving we get

$$\hat{\theta}_S = \int_0^\infty \frac{z^{n+d-1}}{\Gamma(n+d-1)}\theta^{-(n+d)}e^{-(z/\theta)}d\theta$$

$$= \frac{z^{n+d-1}}{\Gamma(n+d-1)}\int_0^\infty \theta^{-(n+d)}e^{-(z/\theta)}\,d\theta$$

$$= \frac{z^{n+d-1}}{\Gamma(n+d-1)}\frac{\Gamma(n+d-2)}{z^{n+d-2}}$$

$$\hat{\theta}_S = \frac{z}{n+d-2}\ ; \text{n+d>2} . \quad (2.2.4)$$

The Bayes estimator under precautionary loss function, say $\hat{\theta}_P$, *using the value of* $f(\theta|\underline{x})$ from equation (2.2.2) is the solution of equation (2.1.11) given by

$$\hat{\theta}_P = [E_\pi(\theta^2)]^{\frac{1}{2}}$$

$$= \left[\int_0^\infty \theta^2 f(\theta|\underline{x})\,d\theta\right]^{\frac{1}{2}}$$

$$= \left[\frac{z^{n+d-1}}{\Gamma(n+d-1)}\int_0^\infty \theta^{-(n+d-2)}e^{-(z/\theta)}d\theta\right]^{\frac{1}{2}}$$

On simplification which leads to

$$\hat{\theta}_P = \frac{z}{[(n+d-2)(n+d-3)]^{\frac{1}{2}}} \quad (2.2.5)$$

3. Bayes Estimator Under $g_2(\theta)$

Under $g_2(\theta)$, the posterior distribution is defind by

$$f(\theta/\underline{x}) = \frac{f((\underline{x}|\theta))g_2(\theta)}{\int_0^\infty f((\underline{x}|\theta))g_2(\theta)d\theta} \quad (2.3.1)$$

Substituting the values of $g_2(\theta)$ and $f(\underline{x}/\theta)$ from equations (2.1.10) and (2.1.7) in (2.3.1) and simplifying, we get

$$f(\theta|\underline{x}) = \frac{\left(\frac{1}{\theta}\right)^n \left(\frac{1}{\prod_{i=1}^n x_i}\right) e^{-\left(\frac{z}{\theta}\right)} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\theta}\right)^{\alpha+1} e^{-\beta/\theta}}{\int_0^\infty \left(\frac{1}{\theta}\right)^n \left(\frac{1}{\prod_{i=1}^n x_i}\right) e^{-\left(\frac{z}{\theta}\right)} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\theta}\right)^{\alpha+1} e^{-\beta/\theta} d\theta}$$

$$= \frac{(\beta+z)^{n+\alpha}}{\Gamma(n+\alpha)} \theta^{-(n+\alpha+1)} e^{-\frac{1}{\theta}(\beta+z)} \quad (2.3.2)$$

The Bayes estimator under squared error loss function is the posterior mean given by

$$\hat{\theta}_s = \int_0^\infty \theta \, f(\theta|\underline{x}) d\theta \quad (2.3.3)$$

Substituting the values of f $(\theta/\underline{x})$from equation (2.3.2) in equation (2.3.3) and on solving, we get

$$\hat{\theta}_s = \frac{(\beta+z)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^\infty \theta^{-(n+\alpha)} e^{-\frac{1}{\theta}(\beta+z)} d\theta$$

$$= \frac{(\beta+z)^{n+\alpha}}{\Gamma(n+\alpha)} \frac{\Gamma(n+\alpha-1)}{(\beta+z)^{n+\alpha-1}}$$

$$\hat{\theta}_s = \frac{\beta+z}{(n+\alpha-1)} \quad (2.3.4)$$

The Bays estimator under precautionary loss function using the value of f $(\theta/\underline{x})$from equation (2.3.2) is the solution of equation (2.1.11) given by

$$\hat{\theta}_P = [E_\pi(\theta)^2]^{\frac{1}{2}} = \left[\int_0^\infty \theta^2 f(\theta|\underline{x}) d\theta\right]^{\frac{1}{2}}$$

$$= \left[\frac{(\beta+z)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^\infty \theta^{-(n+\alpha-1)} e^{-\frac{1}{\theta}(\beta+z)} d\theta\right]^{\frac{1}{2}}$$

which on simplification, leads to

$$\hat{\theta}_P = \frac{\beta+z}{[(n+\alpha-1)(n+\alpha-2)]^{1/2}} \quad (2.3.5)$$

Bayes Estimator Under $g_3(\theta)$

Under $g_3(\theta)$, the posterior distribution is defined by

$$f(\theta|\underline{x}) = \frac{f((\underline{x}|\theta)) g_3(\theta)}{\int_0^\infty f((\underline{x}|\theta)) g_3(\theta) d\theta} \quad (2.4.1)$$

Substituting the values of $g_3(\theta)$ *and* $f(\theta|\underline{x})$from equations (2.1.11) and (2.1.7) in (2.4.1) we get, after simplifying, we get

$$f(\theta|\underline{x}) = \frac{\left(\frac{1}{\theta}\right)^n \left(\frac{1}{\prod_{i=1}^n x_i}\right) e^{-\left(\frac{z}{\theta}\right)} \frac{1}{(\beta-\alpha)}}{\int_\alpha^\beta \left(\frac{1}{\theta}\right)^n \left(\frac{1}{\prod_{i=1}^n x_i}\right) e^{-\left(\frac{z}{\theta}\right)} \frac{1}{(\beta-\alpha)} d\theta}, = \frac{z^{n-1}\theta^{-n} e^{-z/\theta)}}{I_g\left(\frac{z}{\alpha}, n-1\right) - \left(\frac{z}{\beta}, n-1\right)}, \quad (2.4.2)$$

Where

$I_g(x,n) = \int_0^x e^{-x} x^{n-1}$dt is the incomplete gama function

The Bayes estimator under squared error loss function is the posterior mean given by

$$\hat{\theta}_S = \int_\alpha^\beta \theta \, f(\theta|\underline{x}) d\theta \quad (2.4.3)$$

Substituting the values of $f(\theta|\underline{x})$ from equation (2.4.2) in equation (2.4.3) , we get

$$\hat{\theta}_S = \int_\alpha^\beta \theta \frac{z^{n-1}\theta^{-n} e^{-z/\theta}}{I_g\left(\frac{z}{\alpha}, n-1\right) - I_g\left(\frac{z}{\beta}, n-1\right)} d\theta$$

which on simplification leads to

$$\hat{\theta}_S = \left(\frac{I_g\left(\frac{z}{\alpha}, n-2\right) - I_g\left(\frac{z}{\beta}, n-2\right)}{I_g\left(\frac{z}{\alpha}, n-1\right) - I_g\left(\frac{z}{\beta}, n-1\right)}\right) z. \quad (2.4.4)$$

The Bayes estimator under precautionary loss function using the value of $f(\theta|\underline{x})$ from equation (2.4.2), is the solution of equation (2.1.11) given by

$$\hat{\theta}_P = [E_\pi(\theta)^2]^{\frac{1}{2}}$$

$$= \left[\int_0^\infty \theta^2 f(\theta|\underline{x}) d\theta\right]^{\frac{1}{2}}$$

$$= \left[\frac{z^{n-1}}{I_g\left(\frac{z}{\alpha}, n-1\right) - I_g\left(\frac{z}{\beta}, n-1\right)} \int_\alpha^\beta \theta^{-(n-2)} e^{-z/\theta}\right]^{\frac{1}{2}}$$

$$\text{or, } \hat{\theta}_P = \left[\frac{I_g\left(\frac{z}{\alpha}, n-3\right) - I_g\left(\frac{z}{\beta}, n-3\right)}{I_g\left(\frac{z}{\alpha}, n-1\right) - I_g\left(\frac{z}{\beta}, n-1\right)}\right]^{\frac{1}{2}} z. \quad (2.4.5)$$

The equations (2.4.4) and (2.4.5), can be solved numerically.

References

1. Abbas, A.E. (2003). Entropy methods for univariate distributions in decision analysis. Bayesian Inference and Maximum Entropy Methods in Science and Engineering: 22nd International Workshop, edited by C.J. Williams, American Institute of Physics.
2. Basu, D(1955) : "A note on the structure of a stochastic model considered by V.M. Dandekar", Sankhya series B-15. Pp 251-252.
3. Clyde, M. and E.I. George (2003). Model uncertainty. Technical report #2003-16, Statistical and Applied Mathematical Sciences Institute.
4. Dunson, D.B. and Stanford, J.B. (2005) Bayesian inferences on predictors of conception probabilities. Biometrics, 61, 126-133.
5. Ebrahimi, K. (1995). Accounting policies for costs in manufacturing companies. MA Dissertation, Tarbiat Modarres University
6. Gini, c. (1924) : "Premiers researchers surla econlabilite de laemme", Proceedings of the international Mathematic Congress, Toronto, pp 889-892.
7. Henary, L (1973) : "Fundements theoriques des measures de la feconolite naturelle," Reveve del' Institute international de statistique, vol 21, pp 135-151.
8. Mukheerji, S. P. and Islam, R. (1983). A finite range distribution of failures times, Naval Research Logistics Quarterly, Vol. 30, p. 487 – 491.
9. Singh S.N. (1964 a) : " On the time of first birth" , Sankhya, vol26(B) , pp 95-102
10. Vincent, P. (1961): "Researches sua la Fecodite-Biologique Institute National 'D' Etudes Dempgraphic Paris" ,Farnce, Press Universitaries De France.
11. Varian Hal R. (1975) Two Problems in the Theory of Fairness.
12. Zellner, A. (1986) On assessing prior distributions and Bayesian regression analysis with g-prior distributions. In Bayesian Inference and Decision Techniques: Essays in Honor of Bruno de
13. Finetti, (eds. P. K. Goel and A. Zellner), pp. 233{243. North-Holland/Elsevier.