International Journal of Statistika and Mathematika, ISSN: 2277- 2790 E-ISSN: 2249-8605, Volume 9, Issue 3, 2014 pp 85-93

On Exact Moments of Lower Generalized Order Statistics from a Class of Exponential Distributions and its Characterization

Jagdish Saran, Kamal Nain

Department of Statistics, University of Delhi, Delhi -110007, INDIA.

Corresponding Address:

jagdish_saran52@yahoo.co.in

Research Article

Abstract: In this paper, we establish exact expressions for single and product moments of lower generalized order statistics from a family of exponential distributions. Further, on using the expression for conditional moments of lower generalized order statistics, we obtain the characterization for the above class of exponential distributions. For a particular case, these results verify the corresponding results of Khan et al. (2012).

Keywords: Lower generalized order statistics, order statistics, lower record values, single and product moments, family of exponential distributions, exponentiated Weibull distribution, characterization.

1. Introduction

Generalized order statistics (GOS) have been introduced and extensively studied in Kamps (1995 a,b) as a unified theoretical set-up which contains a variety of models of ordered random variables with different interpretations. Examples of such models are: Ordinary order statistics, Sequential order statistics, Progressive type II censored order statistics, Record values, k^{th} record value and Pfeifer's records. There is no natural interpretation of generalized order statistics in terms of observed random samples but these models can be effectively applied in life testing and reliability analysis, medical and life time data, and models related to software reliability analysis. The common approach makes it possible to define several distributional properties at once. The structural similarities of these models are based on the similarity of their joint density function. For the case , when F(.) is an inverse distribution function, we need the concept of lower generalized order statistics, which was first introduced by Pawlas and Szynal (2001) as described below in the next section.

2. Lower Generalized Order Statistics

Let $\{X_n, n \geq 1\}$ be a sequence of absolutely continuous, independent and identically distributed random variables with cdf $F(x) = P(X \leq x)$ and pdf f(x). Assume k > 0, n$\in$ { 2,3,…}, $\tilde{m} = (m_1, m_2, ..., m_{n-1}) \in R^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + (n-r)(m+1) > 0$ for all $r \in \{1, 2, ..., n-1\}$. Then $X^*(r, n, \tilde{m}, k)$, r = 1, 2,…,n , are called Lower Generalized Order Statistics (LGOS) if their joint pdf is given by

$$f^{X^*(1,n,\tilde{m},k), X^*(2,n,\tilde{m},k), ..., X^*(n,n,\tilde{m},k)}(x_1, x_2, ..., x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(F(x_i))^{m_i} f(x_i)\right)(F(x_n))^{k-1} f(x_n), \quad (2.1)$$

where $F^{-1}(0+) > x_1 \geq x_2 \geq ... \geq x_n > F^{-1}(1)$ and $X^*(0, n, \tilde{m}, k) = 0.$

By choosing appropriate values of parameters, we get the distribution of a few very common statistics as shown in Table 2.1 given below.

Table 2.1

S.No.	Choice of parameters for	LGOS becomes
1	$m = 0$ and $k = 1$	$X^*(r, n, m, k) = X_{n-r+1:n}, (n-r+1)^{th}$ order statistics
2	$m = -1$	$X^*(r, n, m, k)$ r^{th} lower k record value

The joint pdf of first r, LGOS is given by :

$$f^{X^*(1,n,\tilde{m},k),X^*(2,n,\tilde{m},k),\ldots,X^*(r,n,\tilde{m},k)}(x_1,x_2,\ldots,x_r)=c_{r-1}\left(\prod_{i=1}^{r-1}(F(x_i))^{m_i}f(x_i)\right)(F(x_r))^{k+n-r+M_r-1}f(x_r), \tag{2.2}$$

where $F^{-1}(0+)>x_1\geq x_2\geq\ldots\geq x_r>F^{-1}(1)$.

We now consider two cases:

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$

Case II: $\gamma_i \neq \gamma_j$; $i\neq j$, i , $j = 1, 2, \ldots, n - 1$.

For case I, the LGOS will be denoted by $X^*(r, n, m, k)$. The pdf of $X^*(r, n, m, k)$ is given by

$$f^{X^*(r,n,m,k)}(x)=\frac{c_{r-1}}{(r-1)!}(F(x))^{\gamma_r-1}f(x)\,g_m^{r-1}(F(x)),\quad x\in R, \tag{2.3}$$

and the joint pdf of $X^*(r, n, m, k)$ and $X^*(s, n, m, k)$, $1\leq r<s\leq n$, is given by :

$$f^{X^*(r,n,m,k),X^*(s,n,m,k)}(x,y)=\frac{c_{s-1}}{(r-1)!\,(s-r-1)!}\Big(\big(F(x)\big)^m f(x)\Big)\,g_m^{r-1}(F(x))$$

$$\cdot\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}(F(y))^{\gamma_s-1}f(y)\quad, x>y, \tag{2.4}$$

where

$$c_{r-1}=\prod_{j=1}^{r}\gamma_j,\quad \gamma_j=k+(n-j)(m+1),\ r=1,2,\ldots,n,$$

$$g_m(x)=h_m(x)-h_m(0),\ x\in(0,1)\ \text{and}$$

$$h_m(x)=\begin{cases}-\dfrac{x^{m+1}}{m+1}, & m\neq-1,\\ -\log x, & m=-1.\end{cases} \tag{2.5}$$

For case II, the LGOS will be denoted by $X^*(r,n,\tilde{m},k)$. The pdf of $X^*(r,n,\tilde{m},k)$ is given by

$$f^{X^*(r,n,\tilde{m},k)}(x)=c_{r-1}f(x)\sum_{i=1}^{r}a_i(r)(F(x))^{\gamma_i-1},\quad x\in R, \tag{2.6}$$

and the joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1\leq r<s\leq n$, is given by

$$f^{X^*(r,n,\tilde{m},k),X^*(s,n,\tilde{m},k)}(x,y)=c_{s-1}\left\{\sum_{i=r+1}^{s}a_i^r(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i}\right\}\left\{\sum_{i=1}^{r}a_i(r)(F(x))^{\gamma_i}\right\}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}, \tag{2.7}$$

where $c_{s-1}=\prod_{j=1}^{s}\gamma_j$, $\gamma_j=k+n-j+M_j$, $s=1,2,\ldots,n$.

Further, it can be proved that

(i) $$a_i(r)=\prod_{j(\neq i)=1}^{r}(\gamma_j-\gamma_i)^{-1},\quad 1\leq i\leq r\leq n$$

(ii) $$a_i^r(s)=\prod_{j(\neq i)=r+1}^{s}(\gamma_j-\gamma_i)^{-1},\quad r+1\leq i\leq s\leq n$$

(iii) $$a_i(r)=(\gamma_{r+1}-\gamma_i)\,a_i(r+1)$$

(iv) $$c_r=c_{r-1}\gamma_{r+1}$$

(v) $$\sum_{i=1}^{r+1}a_i(r+1)=0.$$

The moments of order statistics have generated considerable interest in the recent years. Several recurrence relations and identities satisfied by single as well as product moments of order statistics have been obtained by various authors in the past. These relations help in reducing the quantum of computations involved. Joshi (1978, 1982) established recurrence relations for exponential distribution with unit mean and were further extended by Balakrishnan and Joshi (1984) for doubly truncated exponential distribution. For linear-exponential distribution, Balakrishnan and Malik (1986) derived the similar type of relations which were extended to doubly truncated linear-exponential distribution by Mohie El-Din et al. (1997) and Saran and Pushkarna (1999). Nain (2010 a, b) obtained recurrence relations for ordinary order statistics and k^{th} record values from p^{th} order exponential and generalized Weibull distributions, respectively. The recurrence relations for the moments of generalized order statistics based on non identically distributed random variables were developed by Kamps (1995 a, b). Pawlas and Szynal (2001) obtained recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto and Burr distributions. Saran and Pandey (2004, 2009) established recurrence relations for single and product moments of generalized order statistics from linear-exponential and Burr distributions. Saran and Pandey (2011) obtained recurrence relations for marginal and joint moment generating functions of dual (lower) generalized order statistics from inverse Weibull distribution. In this paper, we derive exact expressions for single and product moments of LGOS for a class of exponential distributions defined below in Section 3, and discuss its various particular cases. Also, we give the characterization of this class of distributions by considering the conditional moments of LGOS. The results so obtained are generalized versions of the corresponding results of Khan et al. (2012).

3. Family of Exponential Distributions

Consider a family of exponential distributions defined by the function

$$F(x) = \left(1-e^{-\Psi(x)}\right)^{\eta}, \eta > 0 \text{ and } 0 < x < \infty, \tag{3.1}$$

where $\Psi(0)=0, \Psi(\infty)=\infty$ and $\Psi(x)$ is monotonic in nature with inverse function $\phi(x)$, i.e., $\psi^{-1}=\phi$. The Table 3.1 given below demonstrates a few standard distributions obtained from (3.1) by choosing appropriate value of the parameter η and the function $\psi(x)$.

Table 3.1

S.No.	Choice of parameter η and the function $\psi(x)$	Family of exponential distribution represents
1	$\eta=1,\ \psi(x)=\lambda x,\ \phi(x)=\frac{x}{\lambda}\ \ 0<x<\infty$ and $\lambda>0.$	Exponential distribution
2	$\eta=1,\ \psi(x)=\lambda x^{n},\ \phi(x)=\left(\frac{x}{\lambda}\right)^{\frac{1}{n}}\ \ 0<x<\infty$ and $\lambda>0.$	Weibull distribution
3	$\eta=1,\ \psi(x)=\mu+\lambda x,\ \phi(x)=\frac{x-\mu}{\lambda},\mu<x<\infty$ and $\lambda>0.$	Linear- exponential distribution
4	$\psi(x)=(\lambda x)^{\eta},\ \phi(x)=\frac{x^{\frac{1}{\eta}}}{\lambda},0<x<\infty$ and $\eta,\lambda>0$	Exponentiated Weibull distribution

The mathematical form of the distribution, as given in (3.1), is very useful for deriving the exact expressions for the single and product moments of LGOS.

Notations

For n = 1, 2, 3,... , a > 0, b > 0, c > 0, $1\le r<s\le n,\ k\ge 1$ and $u,v\in\{0,1,2,...\}$, we denote by

(a) $$H_u(a,b)=\int_0^{\infty} x^u (F(x))^a f(x) g_m^b (F(x))\,dx \tag{3.2}$$

(b) $$H_{u,v}(a,b,c)=\int_0^{\infty}\int_0^{x} x^u y^v (F(x))^a f(x)\left[h_m(F(y))-h_m(F(x))\right]^b (F(y))^c f(y)\,dy\,dx \tag{3.3}$$

(c) $$\mu_{m,n,k}^{u}(r)=E\left(X^*(r,n,m,k)\right)^u \tag{3.4}$$

(d) $$\mu_{m,n,k}^{u,v}(r,s)=E\left(\left(X^*(r,n,m,k)\right)^u\left(X^*(s,n,m,k)\right)^v\right) \tag{3.5}$$

(e) $$\mu_{\tilde{m},n,k}^{u}(r)=E\left(X^*(r,n,\tilde{m},k)\right)^u \tag{3.6}$$

(f) $$\mu_{\tilde{m},n,k}^{u,v}(r,s)=E\left(\left(X^*(r,n,\tilde{m},k)\right)^u\left(X^*(s,n,\tilde{m},k)\right)^v\right) \tag{3.7}$$

4. The Main Lemma

In this section, we derive some results which will be useful later for establishing the main results.

Lemma 4.1 For the class of distributions defined in (3.1) and non-negative finite integers i, j, a, b and c,

(i)

$$H_i(a,b)=\begin{cases}\dfrac{1}{(m+1)^b}\displaystyle\sum_{d=0}^{b}\sum_{w=0}^{\infty}\binom{b}{d}(-1)^d\dfrac{\beta_w(i)}{\frac{w+i}{\eta}+a+(m+1)d+1}, & m\neq -1,\\ b!\displaystyle\sum_{w=0}^{\infty}\dfrac{\beta_w(i)}{\left(\frac{w+i}{\eta}+a+1\right)^{b+1}}, & m=-1,\end{cases}$$

where $\beta_w(i)$ is the co-efficient of t^{w+i} in $\left[\sum_{w=0}^{\infty}\frac{\phi_w(0)}{w!}\left(\sum_{s=1}^{\infty}\frac{t^s}{s}\right)^w\right]^i$ and $\phi=\psi^{-1}$ as defined earlier.

(ii)

$$H_{i,j}(a,b,c)=\begin{cases}\dfrac{1}{(m+1)^b}\displaystyle\sum_{v=0}^{b}\sum_{w=0}^{\infty}\sum_{w'=0}^{\infty}\binom{b}{v}(-1)^v\times\dfrac{\beta_w(i)}{\frac{w+w'+i+j}{\eta}+(a+c+b(m+1)+2)}\\ \qquad\times\dfrac{\beta_{w'}(j)}{\frac{w'+j}{\eta}+((b-v)(m+1)+c+1)}, & m\neq -1,\\ b!\displaystyle\sum_{w=0}^{\infty}\sum_{w'=0}^{\infty}\dfrac{\beta_w(i)}{\left(\frac{w+w'+i+j}{\eta}+a+c+2\right)^{b+1}}\times\dfrac{\beta_{w'}(j)}{\left(\frac{w'+j}{\eta}+a+1\right)^{b+1}}, & m=-1.\end{cases}$$

Proof of (i) .

Case 1. $m\neq -1$

Substituting $g_m^b(F(x))=\left(\dfrac{1-(F(x))^{m+1}}{m+1}\right)^b=\dfrac{1}{(m+1)^b}\displaystyle\sum_{d=0}^{b}\binom{b}{d}(-1)^d(F(x))^{(m+1)d}$ in (3.2), we get

$$H_i(a,b)=\frac{1}{(m+1)^b}\sum_{d=0}^{b}\binom{b}{d}(-1)^d\int_0^{\infty}x^i(F(x))^{a+(m+1)d}f(x)dx.$$

Putting $t=(F(x))^{\frac{1}{\eta}}$, we have

$$H_i(a,b)=\frac{\eta}{(m+1)^b}\sum_{d=0}^{b}\sum_{w=0}^{\infty}\binom{b}{d}(-1)^d\beta_w(i)\int_0^1 t^{w+i+(a+(m+1)d+1)\eta-1}dt$$

$$=\frac{\eta}{(m+1)^b}\sum_{d=0}^{b}\sum_{w=0}^{\infty}\binom{b}{d}(-1)^d\frac{\beta_w(i)}{w+i+(a+(m+1)d+1)\eta},$$

which leads to the relation as stated in (i) for the case $m\neq -1$.

Case 2. $m=-1$

By using repeatedly the combinatorial identity [Ruiz (1996)]

$$\sum_{d=0}^{b}\binom{b}{d}(-1)^d d^k=\begin{cases}0, & k=0,1,2,...,b-1,\\ (-1)^b b!, & k=b,\end{cases}\tag{4.1}$$

we get

$$H_i(a,b)=\lim_{m\to -1}\sum_{w=0}^{\infty}\beta_w(i)\frac{\sum_{d=0}^{b}\binom{b}{d}(-1)^d\times\left(\frac{w+i}{\eta}+a+(m+1)d+1\right)^{-1}}{(m+1)^b}\to\frac{0}{0}$$

$$=\sum_{w=0}^{\infty}\beta_w(i)\left(\frac{w+i}{\eta}+a+1\right)^{-b-1}\times\sum_{d=0}^{b}\binom{b}{d}(-1)^{d+b}\times d^b$$

(by using L-Hospital Rule),

which again on using (4.1) for k = b leads to the relation as stated in (i) for the case m = -1.

Proof of (ii) .

Case 1. $m\neq -1$

From (3.3), we have, for b = 0,

$$H_{i,j}(a,0,c)=\int_0^{\infty}\int_0^{x} x^i y^j (F(x))^a f(x)(F(y))^c f(y)\,dy\,dx$$

$$=\int_0^{\infty} x^i (F(x))^a f(x) G(x)\,dx, \tag{4.2}$$

where

$$G(x)=\int_0^{x} y^j (F(y))^c f(y)\,dy \tag{4.3}$$

Putting $t=(F(y))^{\frac{1}{\eta}}$ in (4.3), we have

$$G(x)=\eta\int_0^{(F(x))^{\frac{1}{\eta}}}\left(\sum_{w'=0}^{\infty}\beta_{w'}(j)\,t^{j+w'}\right)t^{\eta c+\eta-1}dt$$

$$=\eta\sum_{w'=0}^{\infty}\beta_{w'}(j)\int_0^{(F(x))^{\frac{1}{\eta}}} t^{w'+j+\eta(1+c)-1}dt=\sum_{w'=0}^{\infty}\beta_{w'}(j)\left[\frac{(F(x))^{\frac{w'+j}{\eta}+c+1}}{\frac{w'+j}{\eta}+c+1}\right],$$

which on substituting in (4.2) gives

$$H_{i,j}(a,0,c)=\sum_{w=0}^{\infty}\sum_{w'=0}^{\infty}\beta_w(i)\beta_{w'}(j)\frac{1}{w'+j+(c+1)\eta}\int_0^{1} t^{w+w'+i+j+\eta(a+c+2)-1}\,dt$$

$$=\eta^2\sum_{w=0}^{\infty}\sum_{w'=0}^{\infty}\frac{\beta_w(i)}{w+w'+i+j+(a+c+2)\eta}\times\frac{\beta_{w'}(j)}{w'+j+(c+1)\eta}. \tag{4.4}$$

Further, on substituting the value of

$$\left[h_m(F(y))-h_m(F(x))\right]^b=\left[\frac{(F(x))^{m+1}-(F(y))^{m+1}}{(m+1)}\right]^b$$

$$=\frac{1}{(m+1)^b}\sum_{v=0}^{b}\binom{b}{v}(-1)^v (F(x))^{v(m+1)}(F(y))^{(b-v)(m+1)}$$

in (3.3), we get

$$H_{i,j}(a,b,c)=\frac{1}{(m+1)^b}\sum_{v=0}^{b}\binom{b}{v}(-1)^v$$

$$\times\left[\int_0^{\infty}\int_0^{x} x^i y^j (F(x))^{a+v(m+1)} f(x)(F(y))^{(b-v)(m+1)+c} f(y)\,dy\,dx\right]$$

$$=\frac{1}{(m+1)^b}\sum_{v=0}^{b}\binom{b}{v}(-1)^v H_{i,j}(a+v(m+1),\,0\,,(b-v)(m+1)+c),$$

(using (4.2))

which on using (4.4) leads to the relation as stated in (ii) for the case $m \neq -1$.

Case 2: $m = -1$.

The proof is similar to the one as used in case 2 of (i).

5. Explicit Expressions For Single and Product Moments

Case I: $m_1 = m_2 = ... = m_{n-1} = m$

Theorem 5.1 For n = 1, 2, 3,... , $1 \le r < s \le n$, $k \ge 1$ and $u, v \in \{0,1,2,...\}$,

(a) $$\mu^{u}_{m,n,k}(r) = \frac{c_{r-1}}{(r-1)!} H_u(\gamma_r - 1, r-1), \tag{5.1}$$

(b) $$\mu^{u,v}_{m,n,k}(r,s) = \frac{c_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}} \sum_{d=0}^{r-1} \binom{r-1}{d} (-1)^d \times H_{u,v}(m+(m+1)d,\, s-r-1,\, \gamma_s - 1), \tag{5.2}$$

where $H_u(\gamma_r - 1, r-1)$ and $H_{u,v}(m+(m+1)d,\, s-r-1,\, \gamma_s - 1)$ are as defined in Lemma 4.1.

Proof of (a): On using (3.4) and (2.3), the u[th] order moment of $X^*(r,n,m,k)$ is given by

$$\mu^{u}_{m,n,k}(r) = \frac{c_{r-1}}{(r-1)!} \int_0^\infty x^u (F(x))^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))\, dx .$$

By using (3.2), we shall derive the relation as stated in (5.1).

Proof of (b): On using (3.5) and (2.4), we have

$$\mu^{u,v}_{m,n,k}(r,s) = \frac{c_{s-1}}{(r-1)!(s-r-1)!} \int_0^\infty \int_0^x x^u y^v \left((F(x))^m f(x) \right) g_m^{r-1}(F(x)) \times \left[h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} (F(y))^{\gamma_s - 1} f(y)\, dy\, dx . \tag{5.3}$$

Substituting

$$g_m^{r-1}(F(x)) = \left(\frac{1-(F(x))^{m+1}}{m+1} \right)^{r-1} = \frac{1}{(m+1)^{r-1}} \sum_{d=0}^{r-1} \binom{r-1}{d} (-1)^d (F(x))^{(m+1)d}$$

in (5.3), we have

$$\mu^{u,v}_{m,n,k}(r,s) = \frac{c_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}} \sum_{d=0}^{r-1} \binom{r-1}{d} (-1)^d \int_0^\infty \int_0^x x^u y^v (F(x))^{m+(m+1)d} f(x) \times \left[h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} (F(y))^{\gamma_s - 1} f(y)\, dy\, dx .$$

The relation (5.2) follows immediately on using (3.3).

Case II: $\gamma_i \neq \gamma_j;\ i \neq j,\ \ i, j = 1, 2, ..., n-1$

Theorem 5.2 For n = 1, 2, 3,... , $1 \le r < s \le n$, $k \ge 1$ and $u, v \in \{0,1,2,...\}$,

(a) $$\mu^{u}_{\tilde{m},n,k}(r) = c_{r-1} \sum_{i=1}^{r} a_i(r) H_u(\gamma_i - 1, 0), \tag{5.4}$$

(b) $$\mu^{u,v}_{\tilde{m},n,k}(r,s) = c_{s-1} \sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{r}(s)\, H_{i,j}(\gamma_i - \gamma_j - 1, 0, \gamma_j - 1), \tag{5.5}$$

where $H_u(\gamma_i - 1, 0)$ and $H_{i,j}(\gamma_i - \gamma_j - 1, 0, \gamma_j - 1)$ are as defined in Lemma 4.1 .

Proof of (a) : The u^{th} order moment of $X^*(r,n,\tilde{m},k)$, on using (3.6) and (2.6), is given by

$$\mu^{u}_{\tilde{m},n,k}(r)=c_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^{\infty} x^u (F(x))^{\gamma_i-1} f(x)dx.$$

After using (3.2), we shall derive the exact expression given in (5.4).

Proof of (b): On employing (3.7) and (2.7), we get

$$\mu^{u,v}_{\tilde{m},n,k}(r,s)=\int_0^{\infty}\int_0^{x} x^u y^v \left[c_{s-1}\left\{\sum_{j=r+1}^{s} a_j^r(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_j}\right\}\left\{\sum_{i=1}^{r} a_i(r)(F(x))^{\gamma_i}\right\}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}\right]dy\,dx$$

$$=c_{s-1}\sum_{i=1}^{r} a_i(r)\sum_{j=r+1}^{s} a_j^r(s)\int_0^{\infty}\int_0^{x} x^u y^v (F(x))^{\gamma_i-\gamma_j-1} f(x)\,F^{\gamma_j-1}(y) f(y)\,dy\,dx,$$

which on using (3.3), leads to the relation (5.5).

6. Characterization

Let $X^*(r, n, m, k)$, r = 1, 2,…,n be the LGOS from a continuous type of distribution with cumulative distribution function F(x) and probability density function f(x). Then, in view of (2.3) and (2.4), the conditional density function of $Y = X^*(s, n, m, k)$ given $X^*(r, n, m, k) = x$, $1 \le r < s \le n$, is

$$f(y\,|\,x)=\sigma\left[1-\left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1}\left(\frac{F(y)}{F(x)}\right)^{\gamma_s-1}\frac{f(y)}{F(x)},\ 0<y<x<\infty, \tag{6.1}$$

where $\sigma=\dfrac{c_{s-1}}{(s-r-1)!\,c_{r-1}\,(m+1)^{s-r-1}}$.

Theorem 6.1 Let X be a non-negative, absolutely continuous type of random variable with distribution function F(x) satisfying the conditions F(0) = 0 and 0 < F(x) < 1. Then a necessary and sufficient condition for

$$E\left((X^*(s,n,m,k))^i\,|\,X^*(r,n,m,k)=x\right)=\sum_{w=0}^{\infty}\beta_w(i)\left(1-e^{-\psi(x)}\right)^{w+i}\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\frac{w+i}{\eta}+\gamma_{r+j}}\right) \tag{6.2}$$

is that

$$F(x)=\left(1-e^{-\psi(x)}\right)^{\eta},\ x>0,\eta>0,$$

where $\psi(x)$ is monotonic function satisfying $\psi o\phi(x)=x$ for some function $\phi(x)$.

Proof.

Condition is sufficient.

On using (6.1), we have

$$E\left(Y^i\,|\,X^*(r,n,m,k)=x\right)=\sigma\int_0^{x} y^i\left(1-\left(\frac{F(y)}{F(x)}\right)^{m+1}\right)^{s-r-1}\left(\frac{F(y)}{F(x)}\right)^{\gamma_s-1}\frac{f(y)}{F(x)}dy$$

$$=\sigma\sum_{w=0}^{\infty}\beta_w(i)\left(1-e^{-\psi(x)}\right)^{w+i}\int_0^{1} u^{\frac{w+i}{\eta}+\gamma_s-1}\left(1-u^{m+1}\right)^{s-r-1}du, \text{ where } u=\frac{F(y)}{F(x)}$$

$$=\sigma\sum_{w=0}^{\infty}\beta_w(i)\left(1-e^{-\psi(x)}\right)^{w+i} B\left(\frac{w+i}{\eta(m+1)}+\frac{\gamma_s}{m+1},s-r\right)\quad \left(\text{By putting } u^{m+1}=v\right)$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{w=0}^{\infty}\beta_w(i)\left(1-e^{-\psi(x)}\right)^{w+i}\prod_{j=1}^{s-r}\left(\frac{w+i}{\eta}+\gamma_{r+j}\right)^{-1},$$

which, on substituting the value of $\frac{c_{s-1}}{c_{r-1}}$, leads to (6.2).

Condition is necessary

Let

$$Z_r(x)=\sum_{w=0}^{\infty}\beta_w(i)\left(1-e^{-\psi(x)}\right)^{w+i}\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\frac{w+i}{\eta}+\gamma_{r+j}}\right). \qquad (6.3)$$

Then it implies that

$$Z_{r+1}(x)-Z_r(x)=\frac{1}{\eta\gamma_{r+1}}\sum_{w=0}^{\infty}\beta_w(i)(w+i)\left(1-e^{-\psi(x)}\right)^{w+i}\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\frac{w+i}{\eta}+\gamma_{r+j}}\right). \qquad (6.4)$$

Differentiating both sides of (6.3) with respect to x, we have

$$Z_r'(x)=\frac{e^{-\psi(x)}\psi'(x)}{1-e^{-\psi(x)}}\sum_{w=0}^{\infty}\beta_w(i)(w+i)\left(1-e^{-\psi(x)}\right)^{w+i}\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\frac{w+i}{\eta}+\gamma_{r+j}}\right). \qquad (6.5)$$

Using (6.4) and (6.5), we get

$$Z_r'(x)=\gamma_{r+1}\left(Z_{r+1}(x)-Z_r(x)\right)\frac{\eta\ e^{-\psi(x)}\psi'(x)}{1-e^{-\psi(x)}}. \qquad (6.6)$$

Also from (6.2), we have

$$\frac{c_{s-1}}{c_{r-1}}\int_0^x y^i\left(\left(F(x)\right)^{m+1}-\left(F(y)\right)^{m+1}\right)^{s-r-1}\left(F(y)\right)^{\gamma_s-1}f(y)\,dy$$

$$=(s-r-1)!(m+1)^{s-r-1}\left(F(x)\right)^{\gamma_{r+1}}Z_r(x). \qquad (6.7)$$

Differentiating both sides with respect to x, we get

$$\frac{c_{s-1}}{c_{r-1}}\left[\int_0^x y^i\left(\left(F(x)\right)^{m+1}(x)-\left(F(y)\right)^{m+1}\right)^{s-r-2}\left(F(y)\right)^{\gamma_s-1}f(y)\,dy\right]$$

$$=\frac{(s-r-2)!\ \ (m+1)^{s-r-2}\left(F(x)\right)^{\gamma_{r+1}-(m+1)}}{f(x)}\left(\gamma_{r+1}f(x)Z_r(x)+F(x)Z_r'(x)\right). \qquad (6.8)$$

On substituting the value of the expression appearing in the rectangular brackets on the L.H.S. of (6.8) from (6.7) by replacing therein r by r +1, we get

$$\frac{c_{s-1}}{c_{r-1}}\left(\frac{c_r}{c_{s-1}}(s-r-2)!\ \ (m+1)^{s-r-2}\left(F(x)\right)^{\gamma_{r+2}}Z_{r+1}(x)\right)$$

$$=(s-r-2)!\ \ (m+1)^{s-r-2}\left(F(x)\right)^{\gamma_{r+2}}\left(\gamma_{r+1}Z_r(x)+\frac{F(x)}{f(x)}Z_r'(x)\right),$$

which on simplification yields

$$\gamma_{r+1}\left(Z_{r+1}(x)-Z_r(x)\right)=\frac{F(x)}{f(x)}Z_r'(x).$$

Then, on using (6.6), we get

$$\frac{f(x)}{F(x)}=\frac{\eta\ e^{-\psi(x)}\psi'(x)}{1-e^{-\psi(x)}}$$

$$\Rightarrow \qquad F(x)=\left(1-e^{-\psi(x)}\right)^{\eta},\quad x>0,\eta>0.$$

Hence the result.

Remark

In Theorem 5.1, if we put $\psi(x)=(\lambda x)^{\eta},\ \phi(x)=\frac{x^{\frac{1}{\eta}}}{\lambda}$ where $0<x<\infty$ and $\eta,\lambda>0$, we verify the results obtained by Khan et al. (2012) for exponentiated Weibull distribution.

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