A New Sequential Non-Parametric Test for the Two Sample Problem

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Research Article

Abstract: A simple sequential non-parametric test for the two-sample problem is proposed. A method of deriving its ASN function for uniform and exponential distribution is given and their adequacy confirmed by simulation. The test is based on the normal approximation to the distribution of U-statistic. We also consider the small sample sensitivity. We restrict ourselves to the Lehmann alternative. The test is found to be performing well even for small sample size.

***Keywords*:** Sequential; non-parametric; two-sample problem; average sample number; Lehmann alternative**.**

1. Introduction

Phatarfod and Sudbury (1988) considered a Wald-type test for the two-sample problem. Unlike in Bradley, Merchant and Wilcoxon (1966) they exploited the normal approximation to the Wilcoxon Mann-Whitney statistic. They first considered the Lehmann alternative and later on wider type of alternatives.

In this paper we consider a Wald-type test for the two-sample problem. We take $H_0 : F(x) = G(x) = F_0(x)$ $H_1 : F(.) = F_0^{k_0}(.)\ \ G(.) = F_0^{\frac{1}{k_0}}(.)$, $(0 < k_0 < 1)$. Here we wish to make a judgment between two new treatments and there is no prior preference involved. There would be a region of indifference $k_0 < k < k_0^{-1}$, where, if the true situation was $G(.) = F^k(.)$, it would not matter which of the two treatments was chosen. We propose the following Sequential test of strength (α, β). Observations are taken from the X and Y populations in pairs. At each stage, the U-statistic

$$U = \frac{1}{\binom{n}{3}\binom{n}{3}} \sum_c h\left(X_{i_1}, X_{i_2}, X_{i_3}; Y_{j_1}, Y_{j_2}, Y_{j_3}\right)$$

Where

$$h(X_1, X_2, X_3; Y_1, Y_2, Y_3) = \begin{cases} 1, & if \quad M(X_1, X_2, X_3) < M(Y_1, Y_2, Y_3) \\ 0, & otherwise \end{cases}$$

$M(X_1, X_2, X_3)$ =Median of X_1, X_2, X_3 and $M(Y_1, Y_2, Y_3)$ =Median of Y_1, Y_2, Y_3 is calculated.

Sampling is continued as long as

$$B < \frac{f_{H_1}(U)}{f_{H_0}(U)} < A \qquad \text{Where} \qquad A = \frac{1-\beta}{\alpha} \qquad \text{and} \qquad B = \frac{\beta}{1-\alpha}$$

And H_0 or H_1 is accepted according as the R.H.S or the L.H.S inequality is the first not satisfied.

2. Mean and Variance of the U-statistic when $F(x) = G(x)$

MEAN of U is given by $E_{H_0}(U) = \frac{1}{2}$.

VARIANCE of U is given by

$$\text{Var (U)} = \frac{1}{\binom{n}{3}^2}\sum_{c=0}^{3}\sum_{d=0}^{3}\binom{3}{c}\binom{m-3}{3-c}\binom{3}{d}\binom{n-3}{3-d}\xi_{c,d}$$

Where, $\xi_{c,d}$ = Covariance between the two kernels wherein cX observations and dY observations are in common between the two kernels.

$$\text{Var (U)} = \frac{1}{\binom{n}{3}\binom{m}{3}}\sum_{c=0}^{3}\sum_{d=0}^{3}\binom{3}{c}\binom{m-3}{3-c}\binom{3}{d}\binom{n-3}{3-d}\xi_{c,d} = \frac{1}{\binom{n}{3}^2}\sum_{c=0}^{3}\sum_{d=0}^{3}\binom{3}{c}\binom{n-3}{3-c}\binom{3}{d}\binom{n-3}{3-d}\xi_{c,d}$$

$$= \frac{1}{\binom{n}{3}^2}\left\{\begin{array}{l}\left\{\frac{(n-3)^2(n-4)^2(n-5)}{2}\right\}\left(\frac{131}{5775}+\frac{131}{5775}\right)+\left\{(n-3)^2(n-4)(n-5)\right\}\left(\frac{97}{2100}+\frac{97}{2100}\right)\\ +\left\{\frac{(n-3)(n-4)(n-5)}{3}\right\}\left(\frac{1}{12}+\frac{1}{12}\right)+\left\{\frac{9(n-3)^2(n-4)^2}{4}\right\}\left(\frac{2323}{56700}\right)\\ +\left\{9(n-3)^2(n-4)\right\}\left(\frac{19}{252}+\frac{19}{252}\right)+\left\{9(n-3)^2\right\}\left(\frac{3}{28}\right)+\left\{3(n-3)(n-4)\right\}\left(\frac{17}{140}+\frac{17}{140}\right)\\ +\left\{3(n-3)\right\}\left(\frac{83}{140}+\frac{83}{140}\right)+\frac{1}{4}\end{array}\right\} \tag{2.1}$$

3. Mean and variance of the U-statistic when $F(x) = F^{k_0}(x)$ and $G(x) = F^{\frac{1}{k_0}}(x)$

MEAN is given by

$$\mu = E(U) = \frac{\left(19k_0^{\ 2}+6\right)}{\left(k_0^{\ 2}+1\right)\left(3k_0^{\ 2}+2\right)\left(2k_0^{\ 2}+3\right)}$$

VARIANCE of U is given by

$$\text{Var(U)} = \frac{1}{\binom{n}{3}^2}\sum_{c=0}^{3}\sum_{d=0}^{3}\binom{3}{c}\binom{m-3}{3-c}\binom{3}{d}\binom{n-3}{3-d}\xi^*_{c,d}$$

Where, $\xi^*_{c,d}$ =Covariance between the two kernels wherein cX observations and dY observations are in common between the two kernels.

$$\text{Var(U)} = \frac{1}{\binom{n}{3}^2}\sum_{c=0}^{3}\sum_{d=0}^{3}\binom{3}{c}\binom{n-3}{3-c}\binom{3}{d}\binom{n-3}{3-d}\xi^*_{c,d}$$

$$= \frac{1}{\binom{n}{3}^2}\left\{\begin{array}{l}\left\{\frac{(n-3)^2(n-4)^2(n-5)}{2}\right\}\left(\xi^*_{1,0}+\xi^*_{0,1}\right)+\left\{(n-3)^2(n-4)(n-5)\right\}\left(\xi^*_{2,0}+\xi^*_{0,2}\right)\\ +\left\{\frac{(n-3)(n-4)(n-5)}{3}\right\}\left(\xi^*_{3,0}+\xi^*_{0,3}\right)+\left\{\frac{9(n-3)^2(n-4)^2}{4}\right\}\left(\xi^*_{1,1}\right)\\ +\left\{9(n-3)^2(n-4)\right\}\left(\xi^*_{1,2}+\xi^*_{2,1}\right)+\left\{9(n-3)^2\right\}\left(\xi^*_{2,2}\right)+\left\{3(n-3)(n-4)\right\}\left(\xi^*_{1,3}+\xi^*_{3,1}\right)\\ +\left\{3(n-3)\right\}\left(\xi^*_{2,3}+\xi^*_{3,2}\right)+\xi^*_{3,3}\end{array}\right\} \tag{3.1}$$

Where, $\xi^*_{1,0}, \xi^*_{0,1}, \xi^*_{2,0}, \xi^*_{0,2}, \xi^*_{3,0}, \xi^*_{0,3}, \xi^*_{1,1}, \xi^*_{1,2}, \xi^*_{2,1}, \xi^*_{2,2}, \xi^*_{1,3}, \xi^*_{3,1}, \xi^*_{2,3}, \xi^*_{3,2}, \xi^*_{3,3}$ are as defined earlier. Details are omitted for the sake of brevity.

4. The Sequential Non-Parametric Test

We have developed the SPRT based on the statistic U using the normal approximation to our statistic U when n is large for testing the simple hypothesis $H_0 : F(x) \equiv G(x)$ $H_1 : F(x) = F^{k_0}(x)$ Vs $H_1 : G(x) = F^{\frac{1}{k_0}}(x)$.

I.e. the continuation region at stage m is given by

$$\log B - \log \frac{\sqrt{Var_{H_0}(U)}}{\sqrt{Var_{H_1}(U)}} < \frac{1}{2}\frac{\left(U - E_{H_0}(U)\right)^2}{Var_{H_0}(U)} - \frac{1}{2}\frac{\left(U - E_{H_1}(U)\right)^2}{Var_{H_1}(U)} < \log A - \log \frac{\sqrt{Var_{H_0}(U)}}{\sqrt{Var_{H_1}(U)}}$$

Where $A = \frac{1-\beta}{\alpha}$ and $B = \frac{\beta}{1-\alpha}$

5. The Average Sample Number

The sample size needed to reach a decision in a sequential or a multiple sampling plan is a random variable N, because at any stage of the experiment the decision to terminate the process depends on the results of the observations made earlier. The distribution of this random variable depends on the true distribution of the observations during the sampling process.

The ASN is given by

E μ(n) = L (μ) log B+ [1-L (μ)] log A/ E μ(z) where μ is the parameter

If E μ(z) =0 then, E (n) =-logAlogB/E (z^2)

Where E (z^2) = (μ_0- μ_1)^2/σ^2 i.e. E (n) = (-log A log B/(μ_0- μ_1)^2)* σ^2

For this test the ASN function was obtained. To derive the ASN function, we note that the ASN function of our test is given by

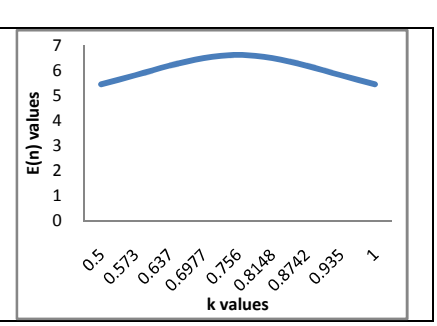
E(N/K)=(L(K)log B+[1-L(K)]log A)/ E(z) Where, $E(z)=\frac{\left[\mu_0^2-\mu_{1-2(\mu_0-\mu_1)*E(x)}^2\right]}{2*\sigma_0^2}$

The following tables and graphs give the ASN for different values of k_0

The values of ASN function for $k_0 = 0.5$ and the corresponding graph.

Table 1

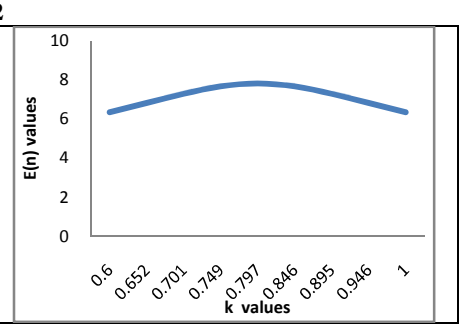
h(μ)	μ	L(μ)	E_μ(n)	K
-1	0.8935	0.04999	5.45	0.5
-0.75	0.8443	0.09901	5.81	0.573
-0.5	0.795	0.1866	6.19	0.637
0.25	0.7459	0.3238	6.49	0.6977
0	0.6967	0.5	6.62	0.756
0.25	0.6475	0.67609	6.49	0.8148
0.5	0.5983	0.81335	6.19	0.8742
0.75	0.5491	0.901	5.81	0.935
1	0.5	0.95	5.45	1



The values of ASN function for $k_0 = 0.6$ and the corresponding graph

Table 2

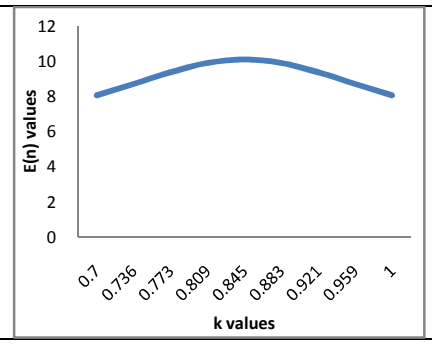
h(μ)	μ	L(μ)	E_μ(n)	K
-1	0.8240	0.04999	6.34	0.6
-0.75	0.7835	0.09901	6.81	0.652
-0.5	0.743	0.1866	7.28	0.701
-0.25	0.7025	0.3238	7.66	0.749
0	0.662	0.5	7.81	0.797
0.25	0.6215	0.67609	7.66	0.846
0.5	0.581	0.81335	7.28	0.895
0.75	0.5405	0.901	6.81	0.946
1	0.5	0.95	6.34	1



The values of ASN function for $k_0 = 0.7$ and the corresponding graph

Table 3

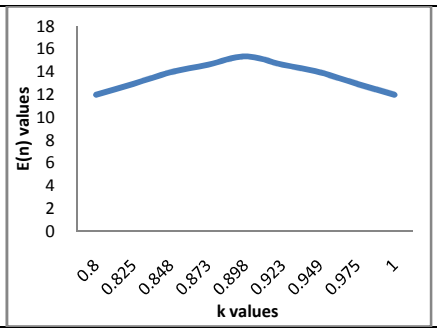
h(μ)	μ	L(μ)	$E_\mu(n)$	K
-1	0.744	0.04999	8.07	0.7
-0.75	0.7135	0.09901	8.72	0.736
-0.5	0.6830	0.1866	9.38	0.773
-0.25	0.6525	0.3238	9.91	0.809
0	0.6220	0.5	10.109	0.845
0.25	0.5915	0.67609	9.91	0.883
0.5	0.5610	0.81335	9.38	0.921
0.75	0.5305	0.901	8.72	0.959
1	0.5	0.95	8.07	1



The values of ASN function for $k_0 = 0.8$ and the corresponding graph

Table 4

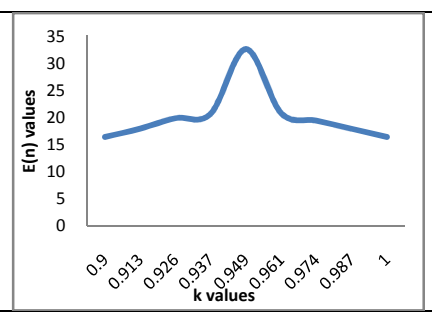
h(μ)	μ	L(μ)	$E_\mu(n)$	K
-1	0.6599	0.04999	11.98	0.80
-0.75	0.6391	0.09901	12.93	0.825
-0.5	0.6192	0.1866	13.94	0.848
-0.25	0.599	0.3238	14.61	0.873
0	0.579	0.5	15.33	0.898
0.25	0.559	0.67609	14.61	0.923
0.5	0.539	0.81335	13.94	0.949
0.75	0.519	0.901	12.93	0.975
1	0.5	0.95	11.98	1



The values of ASN function for $k_0 = 0.9$ and the corresponding graph

Table 5

h(μ)	μ	L(μ)	$E_\mu(n)$	K
-1	0.577	0.04999	16.38	0.90
-0.75	0.5673	0.09901	17.92	0.913
-0.5	0.557	0.1866	19.83	0.926
-0.25	0.5481	0.3238	20.67	0.937
0	0.5385	0.5	32.60	0.949
0.25	0.529	0.67609	20.77	0.961
0.5	0.5192	0.81335	19.39	0.974
0.75	0.5096	0.901	17.89	0.987
1	0.5	0.95	16.39	1



The test has greater ASN for the case K_0=0.9

6. Average Sample Number for Uniform Distribution

We have compared our test with other tests for the same problem. We have considered two distributions, namely Uniform and Exponential distributions. We have found the mean and variance for the two distributions and later found the ASN function.

We have the uniform distribution

$f(x) = 1 \qquad 0 < x < 1$

$g(y) = 1 \qquad -\frac{\delta}{\sqrt{12}} < y < 1 - \frac{\delta}{\sqrt{12}}$

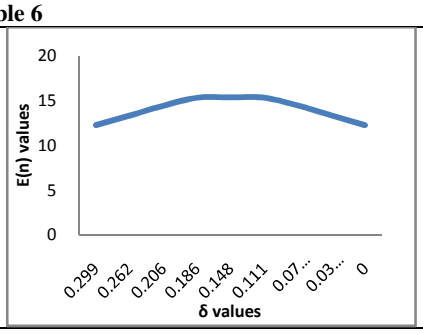
We have obtained the mean and variance of the statistic U under uniform distribution. The calculation is omitted for the sake of brevity.

The ASN for different values of δ

For $\delta = 0.3$

Table 6

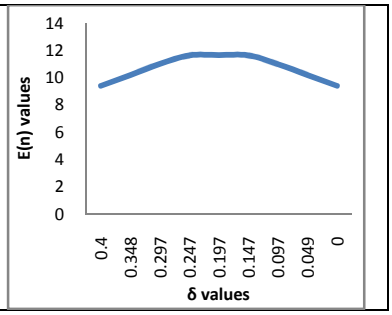
h(μ)	μ	δ	n
-1	0.3973	0.299	12.28
-0.75	0.4101	0.262	13.36
-0.5	0.4229	0.206	14.457
-0.25	0.4358	0.186	15.35
0	0.4487	0.148	15.4
0.25	0.4615	0.111	15.348
0.5	0.4743	0.0742	14.483
0.75	0.4871	0.0372	13.382
1	0.5	0	12.28



$\delta = 0.4$

Table 7

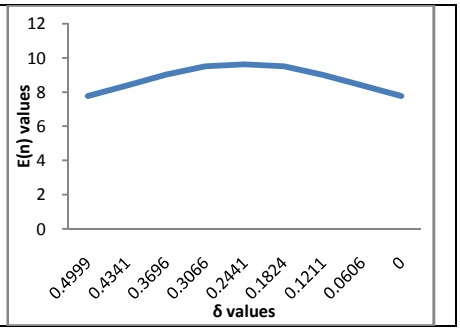
h(μ)	μ	δ	n
-1	0.3642	0.4	9.41
-0.75	0.3812	0.348	10.199
-0.5	0.3982	0.297	11.006
-0.25	0.4151	0.247	11.63
0	0.4321	0.197	11.65
0.25	0.4491	0.147	11.63
0.5	0.4661	0.097	10.99
0.75	0.483	0.049	10.199
1	0.5	0	9.41



For $\delta = 0.5$

Table 8

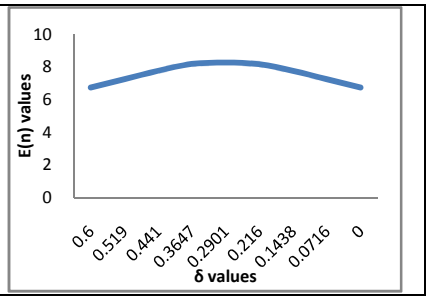
h(μ)	μ	δ	n
-1	0.3322	0.4999	7.77
-0.75	0.532	0.4341	8.39
-0.5	0.3742	0.3696	9.02
-0.25	0.3951	0.3066	9.51
0	0.4161	0.2441	9.62
0.25	0.4371	0.1824	9.51
0.5	0.4581	0.1211	9.01
0.75	0.479	0.0606	8.39
1	0.5	0	7.77



For $\delta = 0.6$

Table 9

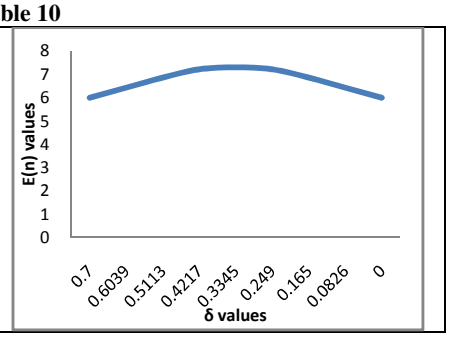
h(μ)	μ	δ	n
-1	0.3012	0.6	6.72
-0.75	0.3261	0.519	7.23
-0.5	0.3509	0.441	7.748
-0.25	0.3758	0.3647	8.167
0	0.4006	0.2901	8.25
0.25	0.4255	0.216	8.15
0.5	0.4503	0.1438	7.747
0.75	0.4752	0.0716	7.23
1	0.5	0	6.72



For $\delta = 0.7$

Table 10

h(μ)	μ	δ	n
-1	0.2715	0.7	6.00
-0.75	0.3	0.6039	6.434
-0.5	0.3286	0.5113	6.87
-0.25	0.3572	0.4217	7.227
0	0.3858	0.3345	7.31
0.25	0.4143	0.249	7.225
0.5	0.4429	0.165	6.87
0.75	0.4714	0.0826	6.434
1	0.5	0	6.00



7. Average Sample Number for Exponential Distribution

We now consider the exponential distribution we have found the mean and variance for the distribution and later found the ASN function.

We have the exponential distribution

$f(x) = e^{-x}$ $\quad 0 < x < \infty$

$g(y) = e^{-(y-\delta)}$ $\quad y \geq \delta$

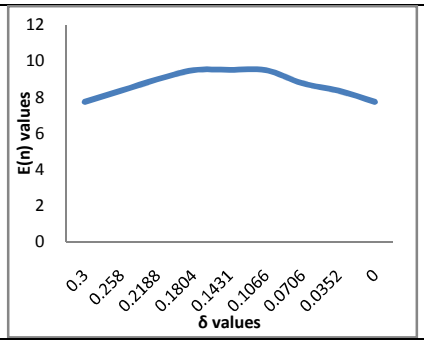
We have obtained the mean and variance of the statistic U under exponential distribution. The calculation is omitted for the sake of brevity.

The ASN for different values of δ

For $\delta\delta = 0.3$

Table 11

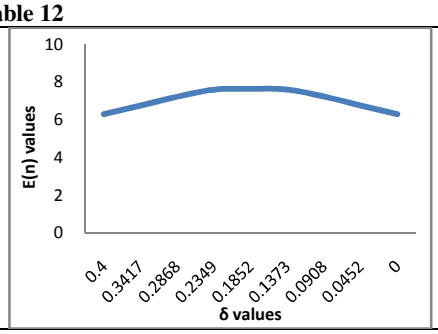
h(μμ)	μμ	δ	n
-1	0.3313	0.3	7.738
-0.75	0.3524	0.258	8.352
-0.5	0.3735	0.2188	8.977
-0.25	0.3946	0.1804	9.48
0	0.4156	0.1431	9.5
0.25	0.4367	0.1066	9.48
0.5	0.4578	0.0706	8.976
0.75	0.4789	0.0352	8.352
1	0.5	0	7.738



For $\delta = 0.4$

Table 12

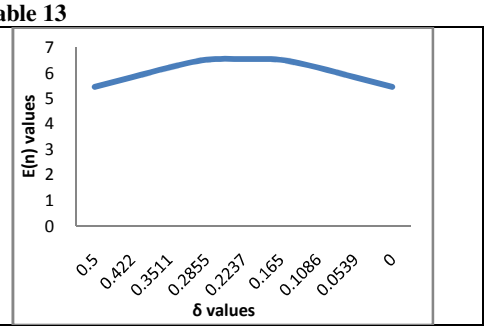
h(μ)	μ	δ	n
-1	0.2839	0.4	6.28
-0.75	0.3109	0.3417	6.739
-0.5	0.3379	0.2868	7.206
-0.25	0.3649	0.2349	7.577
0	0.3919	0.1858	7.62
0.25	0.4189	0.1373	7.587
0.5	0.4459	0.0908	7.212
0.75	0.4729	0.0452	6.742
1	0.5	0	6.28



For $\delta = 0.5$

Table 13

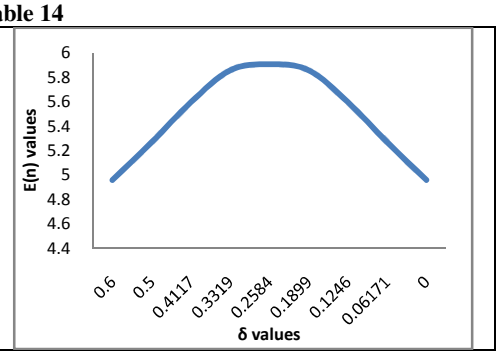
h(μ)	μ	δ	n
-1	0.2418	0.5	5.46
-0.75	0.2741	0.422	5.837
-0.5	0.3064	0.3511	6.218
-0.25	0.3386	0.2855	6.518
0	0.3709	0.2237	6.55
0.25	0.4032	0.1650	6.517
0.5	0.4355	0.1086	6.214
0.75	0.4677	0.0539	5.837
1	0.5	0	5.46



For $\delta = 0.6$

Table 14

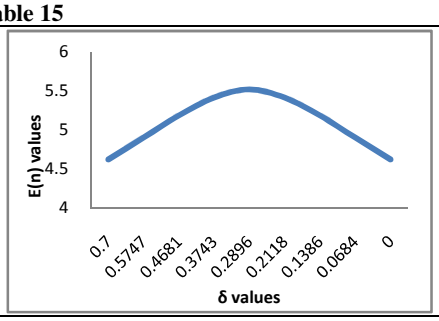
h(μ)	μ	δ	n
-1	0.2049	0.6	4.96
-0.75	0.2418	0.5	5.27
-0.5	0.2787	0.4117	5.598
-0.25	0.3156	0.3319	5.86
0	0.3525	0.2584	5.91
0.25	0.3893	0.1899	5.86
0.5	0.4262	0.1246	5.598
0.75	0.4631	0.06171	5.27
1	0.5	0	4.96



For $\delta = 0.7$

Table 15

h(μ)	μ	δ	n
-1	0.1729	0.7	4.62
-0.75	0.2138	0.5747	4.901
-0.5	0.2547	0.4681	5.186
-0.25	0.2956	0.3743	5.415
0	0.3365	0.2896	5.52
0.25	0.3773	0.2118	5.414
0.5	0.4182	0.1386	5.185
0.75	0.4591	0.0684	4.901
1	0.5	0	4.62



Conclusions

ASN for different values of δ

For $\delta = 0.3$		For $\delta = 0.4$		For $\delta = 0.5$		For $\delta = 0.6$		For $\delta = 0.7$	
Uniform	Exponential	Uniform	Exponential	Uniform	Exponential	Uniform	Exponential	Uniform	Exponential
12.28	7.73	9.41	6.28	7.77	5.46	6.72	4.96	6.00	4.62
13.36	8.35	10.19	6.73	8.39	5.83	7.23	5.27	6.43	4.90
14.45	8.97	11.006	7.206	9.02	6.21	7.74	5.59	6.87	5.18
15.35	9.48	11.63	7.57	9.51	6.51	8.16	5.86	7.22	5.41
15.19	9.32	11.44	7.46	9.36	6.43	8.03	5.77	7.11	5.34
15.34	9.48	11.63	7.58	9.51	6.51	8.15	5.86	7.22	5.41
14.48	8.97	10.99	7.21	9.01	6.21	7.74	5.59	6.87	5.18
13.38	8.35	10.19	6.74	8.39	5.83	7.23	5.27	6.43	4.90
12.28	7.73	9.41	6.28	7.77	5.46	6.72	4.96	6.00	4.62

The results appear in the table. It is seen that for the exponential distribution ASN behaves better than the uniform in having smaller error probabilities and lower ASN, the ASN in general being less than half the fixed sample size for the test of the difference of two means from normal populations with equal known variance.

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